A particle of mass $m$ falls from rest and is subject to a drag force proportional to the square of its velocity $v$. Let $y$ measure the height, with “up” as positive, and use a proportionality constant $b > 0$ to characterize the drag force.  

(a) Write the differential equation of motion for $y = y(t)$. (b) Find the terminal velocity in terms of $m$, $g$, and $b$. (c) The mechanical energy $E = mv^2/2 + mgy$ is not conserved. Find $dE/dt$ in terms of $b$ and $v$. (d) Find the velocity $v = \dot{y}(t)$. You may find it useful that $1/(1-u^2) = (1/2)[1/(1+u) + 1/(1-u)]$. (The algebra is a little hairy; carry it as far as you can to come up with a neat expression for $v$.)
The plot gives recent data taken with the Keck telescope in Hawaii, showing the position of a star, at different times, orbiting the center of our Galaxy. Distance are given in AU, relative to an arbitrary origin, where 1 AU is the distance from the Earth to the Sun. It takes 16 years for one complete orbit.

Use this data to determine, as accurately as you can from the plot, the mass of the object at the center of the Galaxy. (Assume the orbit lies in the plane of the page.) Express your answer as a factor times the mass of the Sun.
A 1 kg mass moves in one dimension $x$ on a horizontal frictionless surface, attached to a spring on one side and through a massless piston to a dashpot on the other side. The spring has stiffness $k = 0.25 \text{ N/m}$ and the dashpot provides a drag force $bv = b\dot{x}$ with $b = 1 \text{ N} \cdot \text{s/m}$. At $t = 0$, the mass is displaced 1 m from its equilibrium position at $x = 0$, and moves with a velocity $-1.5 \text{ m/sec}$. Find the motion $x(t)$, and sketch $x(t)$ over the range $0 \leq t \leq 5 \text{ sec}$. 

You have thirty minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.
Four particles are arranged as shown at the corners of a square of side length $2a$, centered at the origin and lying in the $xy$ plane. The upper right and lower left particles each have mass $3m$, and the other corners have mass $m$.

Find (a) all nine components of the moment of inertia tensor in this coordinate system; the angular momentum vector $\mathbf{L}$ for (b) $\mathbf{\omega} = \omega \hat{z}$ and (c) $\mathbf{\omega} = \omega \hat{x}$; (d) the principle moments of inertia; and (e) the directions of the principle axes. (Partial credit will be given for identifying the principle axes correctly without the accompanying calculation.)
A mass falls from a height $h$ near a point on the Earth’s surface at colatitude $\theta$. It is initially at rest. Air resistance is negligible. Find, to lowest nontrivial order in the Earth’s rotational angular velocity $\Omega$, the North/South deflection in terms of $h$, $\Omega$, $\theta$, and the effective gravitational acceleration $g$. Clearly indicate whether the deflection is to the North or South.
A hoop with radius $R$ lies in the horizontal plane and rotates about point $A$ at fixed angular frequency $\omega$. A bead of mass $m$ moves along the hoop, its position specified by an angle $\phi$ with respect to the center of the hoop, with $\phi = 0$ along diameter $AB$. Find the Lagrangian and then the Lagrange equation of motion for $\phi$.

*Hint:* Find $x$ and $y$ for the mass by first locating the center of the hoop (at angle $\omega t$, with respect to the $x$-axis) and then the bead (at angle $\omega t + \phi$), and then $\dot{x}$ and $\dot{y}$. 

You have thirty minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.
You have thirty minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

A mass $m$ slides without friction, under gravity, on an inclined plane. The equation of the plane is

$$f(x, y) = \frac{x}{l} + \frac{y}{h} = 1$$

Write the Lagrangian $L = L(x, y, \dot{x}, \dot{y}, t)$. Find Lagrange’s (differential) equations of motion for $x$ and $y$ using a Lagrange multiplier $\lambda$, expressing the constraint using the function $f(x, y)$ above.

Then, write the equations of motions for $x$ and $y$ using Newton’s Second Law, incorporating a force $F$ acting on the mass, normal to the surface of the plane. Find an expression for $\lambda$ in terms of $h$, $l$, and the magnitude $N$ of the normal force.
Two masses move horizontally without friction. The smaller mass (5\(m\)) is connected by a spring to a fixed wall, and also to the larger mass (6\(m\)). The larger mass is connected only to the smaller mass. The springs each have stiffness \(k\). Find the eigenfrequencies for this system. Quantitatively describe the amplitudes and motions of the corresponding modes.
A long string with mass $\mu$ per unit length is stretched tightly with a tension $\tau$.  

(a) What is the speed $v$ of wave propagation on the string, expressed in terms of $\mu$ and $\tau$?  

(b) Let $y(x,t)$ be the shape of the string at time $t$, in a coordinate system where $x = 0$ is located somewhere near the middle of the string. Someone comes along at time $t = 0$ and “plucks” the string so that it initially has a shape $y = f(x)$ and is at rest. Find the shape $y(x,t)$ that satisfies the wave equation and the initial conditions, in terms of $v$ and the function $f$.  

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**Quiz #9**  
4 Nov 2011

You have thirty minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.
Note: This quiz has two separate questions. Answer both of them.

(I) The figure shows the periodic response for some nonlinear system as a function of the periodic driving strength. Clearly indicate on the plot, any driving strength that corresponds to

(a) Period two
(b) Period three
(c) Period four
(d) Liapunov exponent $< 0$
(e) Liapunov exponent $> 0$

(II) A particular nonlinear system shows a period “doubling” transition occurs at a driving strength $R = 5$, and another doubling transition, i.e. period “quadrupling”, at $R = 5.5$. Estimate the value of $R$ at which chaos sets in. (If you don’t have a calculator, just indicate what arithmetic you need to do and with what numbers.) You will find it handy to know that $1 + x + x^2 + x^3 + \cdots = 1/(1 - x)$ for a positive real number $x < 1$. 


A particle of mass $m$ moves in two dimensions and is attracted to the origin by a force $\mathbf{F} = -kr$, with $k > 0$. Find the Hamiltonian as a function of the polar coordinates $r$ and $\phi$ and their respective conjugate momenta $p_r$ and $p_\phi$. (You are welcome to just write it down if you know it, but you can also derive it systematically from the Lagrangian if you are not sure.) Explain why $\phi$ is “ignorable”, and rewrite the Hamiltonian as just a function of $r$ and $p_r$ by replacing $p_\phi$ with a constant $\ell$. Then find Hamilton’s equations of motion, and determine the radius $R$, in terms of $\ell$, $m$, and $k$, for circular orbits of the mass $m$. 

A flying saucer orbits the Earth at an altitude well outside the atmosphere. Consider the saucer as a thin disk of mass \( M \) and radius \( R \). Thrusters placed on either side of the station produce a constant torque \( \Gamma \) about the \( \hat{e}_3 \) axis through the center of, and perpendicular to, the disk. If directions “1” and “2” label the other principle axis directions, find the angular velocity vector \( \mathbf{\omega}(t) = \omega_1(t)\hat{e}_1 + \omega_2(t)\hat{e}_2 + \omega_3(t)\hat{e}_3 \) when (a) \( \mathbf{\omega}(0) = 0 \) and (b) \( \mathbf{\omega}(0) = \omega_0\hat{e}_1 \). Express your answer in terms of \( M, R, \omega_0, \) and \( t \). Briefly describe the motion for each case in words. For partial credit, find the answer in terms of the principle moments of inertia for the disk. To find the principle moments in terms of \( M \) and \( R \), it is useful to recall in polar coordinates that the area element is \( \rho d\rho d\phi \), and also that \( \cos^2 \phi = \frac{(1 + \cos 2\phi)}{2} \).
Assume an inviscid fluid with constant density $\rho$ over all space. Consider three different velocity fields for that fluid, namely $v_a(r) = k x \hat{x}$, $v_b(r) = k r \hat{r}$, and $v_c(r) = k r \hat{\phi}$, using usual Cartesian and spherical coordinates, where $k$ is a constant. For each of these velocity fields,

(A) Given a sphere of radius $R$ centered at the origin, find the rate of change of fluid mass contained within the sphere in terms of $\rho$, $k$ and $R$.

(B) For $g = -g \hat{z}$, find the pressure gradient $\nabla p$ in terms of $\rho$, $k$, and $g$. (In the case of $v_c$ this is tricky. You can leave your answer in terms of $\partial \hat{\phi}/\partial \phi$ if you want.)