(1) (See Taylor 13.18.) Some time ago, we showed that the Lagrangian for a mass \( m \) with charge \( q \) moving otherwise freely in a static magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} \) with \( \mathbf{A} = \mathbf{A}(\mathbf{r}) \) is \( L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\mathbf{r} \cdot \mathbf{A} \). Show that the Hamiltonian is \( H(\mathbf{r}, p) = (p - q\mathbf{A})^2/2m \) and that Hamilton’s equations give you the correct equation of motion, namely \( m\ddot{\mathbf{r}} = q\mathbf{r} \times \mathbf{B} \).

(2) (See Taylor 13.25.) Consider a Hamiltonian \( H(q, p) \) with one degree of freedom. Show that the coordinate transformation \( \{q, p\} \rightarrow \{Q, P\} \) where \( q = \sqrt{2P} \sin Q \) and \( p = \sqrt{2P} \cos Q \) is “canonical”, that is \( H(Q, P) \) also satisfies Hamilton’s equations. For the (scaled) simple harmonic oscillator Hamiltonian \( H(q, p) = (q^2 + p^2)/2 \), carry out this coordinate transformation and show that \( Q \) is ignorable. What is \( P \)? Solve Hamilton’s equations in terms of \( Q \) and \( P \), and then transform back to \( q \) and \( p \) and show that this is the expected behavior.

(3) (See Taylor 13.28.) An object of mass \( m \) moves in one dimension \( x \) according to a force \( F = kx \) with \( k > 0 \). Find the potential energy \( U(x) \) with \( U(0) \equiv 0 \), and discuss the types of motion separately for total energy \( E > 0 \) and \( E < 0 \). Now write down the Hamiltonian \( H(x, p) \) and derive phase space trajectories for \( E > 0 \) and \( E < 0 \). Briefly explain the connection between these two ways of describing the motion.

(4) Investigate Poincare sections for our driven damped pendulum, in particular sensitivity to what value with a period at which to draw the plot. Use the standard parameter settings \( \beta = \omega_0/4, \omega_0 = 1.5\omega \), and \( \omega = 2\pi \), with initial conditions \( \phi(0) = -\pi/2 \) and \( \dot{\phi}(0) = 0 \). Choose a “chaotic” driving strength, for example \( \gamma = 1.105 \). Then draw sections for times \( t_0 + dt \) where \( 0 \leq dt \leq 1 \). You can do this with the Manipulate function in Mathematica if you want to see smoothly how the section changes.

(5) (See Taylor 12.23 and 12.24.) This is really a math problem, aiming to show you a little about nonlinear equations and how they behave using a logistics map. Using a calculator, MATLAB, Mathematica, or the computer program of your choice, build a logistics map for the relation \( x_{t+1} = f(x_t) \) where \( f(x) = r \sin(\pi x) \). Take \( t = 1, 2, \ldots, t_{\text{max}} \) with \( t_{\text{max}} = 20 \). Produce maps for the three cases (a) \( r = 0.1 \), (b) \( r = 0.5 \), and (c) \( r = 0.78 \). Contrast the three behaviors; you should notice that the behavior for \( r = 0.78 \) is rather different than the other two. Analyze \( f(x) = r \sin(\pi x) \) to find the values of \( r \) at which a second fixed point appears, and show that this is consistent with (b), and that (c) is in a region where you no longer expect any stable fixed points.