(1) The solution to the equation of motion for the one dimensional simple harmonic oscillator can be written in complex notation as $x(t) = Be^{i\omega t} + Ce^{-i\omega t}$. Explain why, for a physical mass-and-spring oscillator with $k = m\omega^2$, we need to enforce that $C = B^*$, and show that this gives the appropriate number of free parameters for a second order differential equation. Write $B = Ae^{-i\beta}$ for real parameters $A$ and $\delta$, and derive $x(t)$, $\dot{x}(t)$, the kinetic energy $\frac{m\dot{x}^2}{2}$, the potential energy $\frac{kx^2}{2}$, and the total energy in terms of $A$, $\delta$, $m$, and $\omega$. Also find the average kinetic and potential energies.

(2) In class, we separately derived the two solutions (each) for the under damped ($\beta < \omega_0$) and critically damped ($\beta = \omega_0$) oscillator. Here, derive the two critically damped solutions by taking the limit as $\beta \to \omega_0$ of the under damped solutions.

(3) This problem is not about “mechanics”, but uses the same mathematics we’ve addressed in class. A resistor $R$, capacitor $C$, and inductor $L$ are connected in series. An AC voltage source $V(t) = V_0 e^{i\omega t}$ is put across these elements. Find the amplitude of the voltage drop $V_R(\omega)$ across the resistor, after transients have died away, and find the resonant frequency. (Assume the system is weakly damped.) For $R = 400\Omega$, $C = 10\mu F = 10\mu\mu F$, and $L = 2\text{mH}$, make a plot of $V_R(\omega)/V_0$ and also the phase $\delta(\omega)$ versus $\omega$, covering the resonance. [Recall that for a current $I = \dot{q}$ through a circuit element, $V_R = IR$, $V_C = q/C$, and $V_L = L\dot{I}$.]

(4) A damped oscillator starts from rest at the origin at $t = 0$, at which time an oscillating driving force $F(t) = F_0 \cos \omega t$ is applied. The oscillator has a “natural” frequency $\omega_0$ (that is, the frequency it would have if there were no damping) and a damping parameter $\beta$. Find the complete solution, including the initial conditions. In the following, set $F_0/m = 1$ and $\omega_0 = 2\pi$ and plot the position $x(t)$ as a function of time, starting at $t = 0$ up to a large enough time so that the transients have died away. For $\omega = \omega_0$, make plots for (a) $\beta = 0.1\omega_0$ and (b) $\beta = 0.9\omega_0$. Repeat for $\omega = 2\omega_0$. Comment on your results.

(5) (See Taylor Problem 5.53.) An oscillator is driven by the following periodic force:

(a) Find the long-term motion $x(t)$ assuming a natural period $\tau_0 = 2$, damping coefficient $\beta = 0.1$, and drive strength $f_{\text{max}} = 1$. Find the Fourier coefficients in the Fourier series for $x(t)$ and plot the sum of the first four terms in the series for $0 \leq t \leq 6$. (b) Repeat, except with natural period $\tau_0 = 3$. Why is the result so different? What about $\tau_0 = 4$?

You can work the integrals and use whatever computer program you’d like to make the plots. However, you are also welcome to use a symbolic manipulation program like MATHEMATICA or MAPLE to do the work. If you use MATHEMATICA, you would probably want to make use of the built-in functions TriangleWave and FourierCosSeries.