You have two hours to complete this exam. There are a total of four problems and you are to solve all of them. *Not all the problems are worth the same number of points.*

You may use your textbooks and class notes and handouts, or other books. You *may not* share these resources with another student during the test.

*Indicate any figures or tables you use in your calculations. Show all work!*

GOOD LUCK!

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Total Score: ______ 100
Problem 1 (5+5+10=20 points): A pendulum of length $l$ and mass $m$ hangs vertically, at rest. It is free to swing in a complete circle in the vertical plane, should someone make it do so. The mass is struck a sharp blow horizontally, imparting the mass with an initial speed $v_0$. The acceleration due to gravity near the surface of the earth is vertically downward with magnitude $g$.

a) What is the period $T$ of the pendulum, assuming that it swings to small angles? Express your answer only in terms of $v_0$, $l$, $m$, and $g$.

For small oscillations, the period of a pendulum is $T = 2\pi \sqrt{\frac{l}{g}}$.

b) The pendulum reaches a maximum angle $\theta_m$, where $\theta = 0$ is the initial, vertical angle. Determine an equation for $v_0$. Express your answer only in terms of $\theta_m$, $l$, $m$, and $g$.

When the pendulum is struck, the potential is zero and the kinetic energy is $\frac{1}{2} mv_0^2$.

When the pendulum reaches its maximum angle, it isn’t moving any more so its kinetic energy is zero. Its potential energy is $mgl(1 - \cos \theta_m)$.

These two have to be equal, since they both equal the total energy. Therefore

$$\frac{1}{2} mv_0^2 = mgl(1 - \cos \theta_m)$$

$$v_0^2 = 2g l(1 - \cos \theta_m)$$

$$v_0 = \sqrt{2g l(1 - \cos \theta_m)}$$

c) Assuming the small angle approximation, show that the pendulum’s motion is described by

$$\theta = \theta_m \cos (\omega t + \phi)$$

for the appropriate values of $\omega$ and $\phi$. Express $\omega$ and $\phi$ only in terms of $\theta_m$, $l$, $m$, and $g$.

There are various ways to go about this, which we did in class. The simplest is to use the small-angle equation of motion, from second period, Tuesday, Sept. 8:

$$\frac{d^2 \theta}{dt^2} = -\omega^2 \theta \quad \text{for} \quad \omega^2 = g/l$$

The solution to this can be written $\theta = \theta_m \cos (\omega t + \phi)$ but we need $\theta = 0$ at $t = 0$, so $\phi = \pi/2$. 
Problem 2 (5+7+8+10=30 points). A string of total mass $m$ and length $L$ is tied with tension $T$ between two walls as shown here:

A) What is the speed $s$ of a traveling wave on this string? Express your answer only in terms of $m$, $L$, and $T$.

The mass density is $\rho = m/L$ so $s = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{LT}{m}}$.

B) On the figure above, sketch the shape of the third ($n = 3$) normal mode, for a wave with amplitude $A$. Clearly label the amplitude and any other important features.

The $n = 3$ normal mode will have two nodes. The amplitude is simply the highest point of the standing wave.

C) What is the equation $y(x, t)$ for this normal mode. You can quote the answer any way you like, but be sure to show how any parameters you use are related to $s$, $m$, $L$, and $T$.

The equation for the $n = 3$ normal mode is $y(x, t) = A \sin \left( \frac{3\pi x}{L} \right) \cos \left( \frac{3\pi st}{L} \right)$.

D) Show how your equation above for $y(x, t)$ can be written in terms of two traveling waves, one moving leftward and the other moving rightward, both with speed $s$.

A rightward wave will look like $Y(x - st)$ and a leftward wave $Y(x + st)$. A combination of this sort which adds up to the wave above is

$$y(x, t) = \frac{A}{2} \sin \left[ \frac{3\pi}{L} (x - st) \right] + \frac{A}{2} \sin \left[ \frac{3\pi}{L} (x + st) \right]$$

where you prove it using the trigonometric identity

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$
**Problem 3 (15 points):** Calculate the Fourier Transform $A_\delta(k)$ of the Dirac Delta function $\delta(x)$.

Explain, in the best way you can, how the relation $\Delta k \cdot \Delta x \sim 1$ applies in this case.

The Fourier Transform is

$$A_\delta(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} \, dx.$$ 

However, the Dirac Delta function has the property that $\int_{-\infty}^{\infty} \delta(x) F(x) \, dx = F(0)$.

Therefore

$$A_\delta(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}} e^{-ik(0)} = \frac{1}{\sqrt{2\pi}},$$

a constant.

The width $\Delta x$ of the delta function is zero, and the width $\Delta k$ of its Fourier transform is infinite, since the constant function goes on forever. In some fashion, the product of zero times infinity, in this case, is something of order one.
Problem 4 (5+5+15+10=35 points): In the following circuit, \( q(t) \) is the charge on the capacitor and a current \( i(t) \) is flowing through all the components:

![Circuit Diagram]

A battery is attached to the voltage tap between the capacitor and the resistor, but after it is there for a long time, it is quickly disconnected so that at \( t = 0 \) no current flows and \( V(0) = V_0 \). This whole problem is an electrical analog of mechanical stuff we’ve done before.

a) What are the initial conditions \( q(0) \) and \( \dot{q}(0) \)? Express only in terms of \( R, L, C \), and \( V_0 \).

The voltage on the capacitor is \( V_0 \) and no current is flowing, at \( t = 0 \). Therefore the initial conditions are simply \( q(0) = CV_0 \) and \( \dot{q}(0) = 0 \).

b) Write the differential equation that governs \( q(t) \). Include a brief explanation of each term.

The battery is disconnected so that for \( t > 0 \), the voltage \( V(t) = 0 \). Adding up the voltage drops along the string, and setting the sum equal to zero, gives

\[
\frac{1}{C}q + iR + L\frac{di}{dt} = 0
\]

where each term is the voltage drop of one of the three elements. Rearranging,

\[L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C}q = 0\]

This is Eqn. 2.3 in Nettel, with no applied voltage.

c) Show that \( \frac{dE}{dt} = -i^2 R \) where \( E = \frac{1}{2} Li^2 + \frac{1}{2C}q^2 \). Include a brief explanation of the physics behind each of the terms in these two equations.

\[
\frac{dE}{dt} = Li\frac{di}{dt} + \frac{1}{C}q\frac{dq}{dt} = i \left[ L \frac{d^2 q}{dt^2} + \frac{1}{C}q \right] = -i^2 R \text{ where we used the answer to (b).}
\]

The two terms in \( E \) are the energy stored in inductor and capacitor. The change in energy is the power dissipated by the resistor.

d) Write the equation for \( q(t) \) when \( 4L = R^2 C \). Express only in terms of \( R, L, C \), and \( V_0 \).

This is the “critical damping” condition for the electrical oscillator. See Prob.3 on the homework that was due on Sept.1. You can get the solution by substituting the electrical analogs. It is given by

\[q(t) = CV_0 \exp\left(-\frac{Rt}{2L}\right)\left(1 + \frac{Rt}{2L}\right)\].