1) From our last homework problem, you found that the electric field wave vector propagating in the \( z \)-direction could be written in the form
\[
\vec{E} = E(\hat{r}, t) \hat{j} \quad E(\hat{r}, t) = E_0 \sin\left(\frac{\pi x}{a}\right) \exp\left[i(kz - \omega t)\right]
\]
so long as
\[
 k = \left(\frac{\omega}{c}\right)\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{1/2}
\]
where \( \omega_c = \frac{\pi c}{a} \). Here, I use the symbol \( k \) to mean the wave number for the wave that is propagating in the \( z \)-direction. Using this relationship between \( k \) and \( \omega \), determine

a) The phase velocity \( v_{ph} = \frac{\omega}{k} \)

b) The group velocity \( v_g = \frac{d\omega}{dk} \)

c) The product \( v_{ph} \cdot v_g \)

2) Show that if one makes the following simultaneous substitutions, for any function \( \lambda(\hat{r}, t) \),
\[
\vec{A}' = \vec{A} + \vec{\nabla} \lambda \\
\phi' = \phi - \frac{\partial \lambda}{\partial t}
\]
for the electromagnetic potentials, then the electric and magnetic fields \( \vec{E} \) and \( \vec{B} \), which are all that we can observe physically, are completely unchanged.

Extra Credit (worth another 50%): Find the differential equation that relates \( \vec{A} \) and \( \phi \) so that the wave equation for the vector potential can be written in terms of “real” currents only, that is
\[
\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}
\]
This is called the “Lorentz” gauge.