1) In class we showed that the energy density in an electromagnetic field can be written as
\[ u = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 B^2 \]
where the “square” \( A^2 \) of a vector field \( \vec{A} \) is the same as \( \vec{A} \cdot \vec{A} \). Using Maxwell’s Equations and the vector calculus identity \( \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \), show that
\[ \nabla \cdot \vec{S} = \frac{\partial u}{\partial t} \]
where the Poynting Vector \( \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \). Briefly explain why this means that \( \vec{S} \) can be interpreted as an “energy current density” in an electromagnetic wave.

2) This problem describes the simple waveguide we discussed in class. An electromagnetic wave propagates in a TE mode between two parallel plates of infinite extent. The plates are made of perfect conductors, and lie parallel to the \( yz \) plane, and separated by a distance \( \Delta x = a \). The electric field of the wave is represented by
\[ \vec{E} = E(\hat{r}, t) \hat{j} \quad E(\hat{r}, t) = E_0 \sin\left(\frac{\pi x}{a}\right) \exp\left[i(kz - \omega t)\right] \]

a) Explain why this form satisfies the boundary conditions for the electric field.

b) Show that, for \( E \) to satisfy the wave equation \( \nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \), we must have
\[ k = \left(\frac{\omega}{c}\right) \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{1/2} \]
where \( \omega_c = \frac{\pi c}{a} \).

c) Find an expression for the phase velocity \( v_{ph} = \frac{\omega}{k} \)

d) Find an expression for the group velocity \( v_g = \frac{d\omega}{dk} \)

e) Determine the product \( v_{ph} \cdot v_g \)

f) Find the cut-off frequency for plates separated by 10cm, and make a plot of the phase and group velocities of the wave for \( \omega_c \leq \omega \leq 10\omega_c \).