Exercise 2: Taylor Series and Applications to Oscillations

(Due at the start of class, Friday, Sept 10 at 10am)

This exercise will get you familiar with Taylor series and in their use at getting approximate solutions to oscillating systems. You will also get familiar with scaling differential equations prior to solving them. The entire exercise discusses the simple pendulum we discussed in class. See also Section 4.6 of Greene. The questions you are to answer are interspersed in the exercise worksheet below.

We begin with the potential energy of the simple pendulum, derived in class and in many, many textbooks.

\[ U := m g L \left(1 - \cos(\theta)\right) \]

You can show that this potential has a minimum at \( \theta = 0 \), and therefore represents an equilibrium point about which oscillations can happen.

\[ a := \text{solve} \left( \frac{\partial}{\partial \theta} U = 0, \theta \right) \]

\[ a := 0 \]

1) Why does a minimum in the potential energy curve represent an "equilibrium" position?

A Taylor expansion about this minimum reveals that to lowest order, this potential energy curve is indeed quadratic and therefore the motion (for small values of \( \theta \)) is that of simple harmonic motion.

\[ \text{taylor}(U, \theta) \]

\[ \frac{1}{2} m g L \theta^2 - \frac{1}{24} m g L \theta^4 + O(\theta^6) \]

2) Verify using the formula for Taylor coefficients that the above expansion is indeed correct.

Let’s compare the actual potential energy function and the Taylor approximation to it. For this, we should assign values to the parameters.

\[ m := 1 \]

\[ g := 9.8 \]

\[ L := 1 \]

\[ U1 := \frac{m g L \theta^2}{2} \]
The true potential energy function (in red) lies between the two other curves. It is well approximated by the lowest order quadratic, and even better by the quartic, approximations for values near $\theta = 0$, say between -1 and 1. The deviations are pretty significant, though, beyond that point. We will now investigate the effect of these approximations on the oscillation behavior and compare them to the "true" solution, using Maple to solve the differential equations for us. We’ll start from scratch here, using a **restart** command to clear away what we’ve defined so far.

> **restart**

We will start with the "scaled" differential equation of motion, where time $t$ is measured in terms of the fraction of a *period of the lowest order solution*, namely $\tau = \frac{t}{T}$, with $T = 2 \pi \sqrt{\frac{L}{g}}$. This is

$$eq := \frac{\partial^2}{\partial \tau \partial \tau} \theta(\tau) + 4 \pi^2 \sin(\theta(\tau)) = 0$$

The Taylor approximation of the sin function is what gives us normal simple harmonic motion in
lowest order, and also shows us how to form higher order corrections to this motion. (You can also do
this by using the approximations to the energy we derived above.)

\[
\theta - \frac{1}{6} \theta^3 + \frac{1}{120} \theta^5 + O(\theta^6)
\]

This gives the lowest order equation of motion (analogous to the lowest order expression for the
potential energy, above) to be

\[
\frac{d^2}{d\tau^2} \theta(\tau) + 4 \pi^2 \theta(\tau) = 0
\]

The solution to this equation is just the solution for normal simple harmonic motion...

\[
\theta(\tau) = A \cos(2 \pi \tau) + B \sin(2 \pi \tau)
\]

... for arbitrary initial conditions. Notice that this solution indeed has a period from \(\tau = 0\) to \(\tau = 1\). If
we try to solve the exact equation this way, we get

\[
\theta(\tau) = \text{RootOf}
\]

\[
\int_{\text{RootOf}}^{-Z} \frac{1}{\sqrt{8 \pi^2 \cos(c) - 8 \pi^2 \cos(A) + B^2}} d_c - \tau + \int_{\text{RootOf}}^{A} \frac{1}{\sqrt{8 \pi^2 \cos(c) - 8 \pi^2 \cos(A) + B^2}} d_c
\]

which basically means that the solution cannot be found analytically. However, Maple can solve the
problem numerically. This can be done by invoking the "numeric" option in the dsolve function. See
Greene, Sections 1.3.2 and 4.6.2, although I’ll be using a different way of plotting the results in this
exercise. I’ll illustrate this using a particular set of initial conditions and your job will be to extend the
First, solve the approximate solution as we have done before. Store the plot as a "plot structure" by using the assignment form of "plot".

```
> sol1 := rhs(dsolve({eq1, θ(0) = A, D(θ)(0) = B}, θ(τ)))
sol1 := cos(2 π τ)
```

Using "plot" to return a plot structure will result in a long list of uninteresting numbers. Therefore, use the ":" at the end of the input line to suppress the output. This requires that we go out of "standard math" input format. (To do this, just right-click on the input line.)

```
> sol1plot := plot(sol1, τ = 0 .. 4, color=blue, linestyle=3):
```

Now solve the exact equation numerically, and use the "odeplot" function to plot it. You have to invoke the "plots" package for this.

```
> sol := dsolve({eq, θ(0) = A, D(θ)(0) = B}, θ(τ), numeric)
sol := proc(rkf45_x) ... end
```

```
> with(plots)
```

```
animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,
listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,
sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot
```

```
> solplot := odeplot(sol, [τ, θ(τ)], 0 .. 4, color=red, linestyle=1, numpoints=100):
```

```
> display([solplot, sol1plot])
```
3) Which of these two curves has the longer period, and why? Use the potential energy curves we plotted earlier to explain your answer.

4) Repeat this procedure, including answering question "3" above, for the comparison of the same two systems (exact solution and the lowest order approximation) but with initial angles θ equal to (a) 0.5 and (b) 7π/8 radians.

5) Is the solution to the exact solution a sine curve at all? Try starting with an initial condition for the pendulum very close (but not quite at) the very top of the swing.

6) Repeat this procedure, including answering question "3" above, for initial angle θ = 1 radians, but for the next order approximation to the simple pendulum. (That is, use the cubic approximation to the sine function, based on the Taylor series we derived earlier.