## Problem Points Score

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1

For the system shown below, the strut AB transmits an axial compressive force while cables BC and BD transmit axial tensile forces. A vertical load, $P$, having a magnitude of 14 kips, is applied at joint B.

![Diagram of the system](image)

a) Draw a free-body diagram (FBD) of joint B. This FBD must be a separate figure. (6 points).

b) For each force shown in your FBD, express the force in Cartesian vector form (i.e. express the force vector in terms of the unit vector in Cartesian coordinates and the vector magnitude). Be sure to use proper vector notation. (12 points)

c) Using the results from part b, write the scalar equations of equilibrium for the FBD of joint B. (8 points).

d) Solve the equations of equilibrium to determine the magnitude of the compressive force in the strut and the magnitude of the tensile forces in the cables. The method of solution is arbitrary but you need to show your work. (4 points).
Solution:

Part a) 6 pts (4 pts for showing each force with correct orientation, 2 pts for showing correct sense of forces)

$$\vec{P} = -14\hat{j} \text{ kips}$$

2 pts (1 point for unit vector and 1 point for magnitude of force)

$$\hat{e}_{AB} = \frac{\vec{r}_{AB}}{\|\vec{r}_{AB}\|} = \frac{6\hat{i} + 6\hat{j} + 3\hat{k}}{\sqrt{6^2 + 6^2 + 3^2}} = \frac{6\hat{i} + 6\hat{j} + 3\hat{k}}{9} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

$$\vec{F}_{AB} = F_{AB}\hat{e}_{AB} = F_{AB}\left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right) = F_{AB}\left(0.667\hat{i} + 0.667\hat{j} + 0.333\hat{k}\right)$$

4 pts (3 pts for unit vector and one point for magnitude of force)

$$\hat{e}_{BC} = -\hat{i}$$

$$\vec{F}_{BC} = F_{BC}\hat{e}_{BC} = F_{BC}(-\hat{i})$$

2 pts (1 point for unit vector and one point for magnitude of force)

$$\hat{e}_{BD} = \frac{\vec{r}_{BD}}{\|\vec{r}_{BD}\|} = \frac{-6\hat{i} + 4\hat{j} - 12\hat{k}}{\sqrt{6^2 + 4^2 + 12^2}} = \frac{-6\hat{i} + 4\hat{j} - 12\hat{k}}{14} = -\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{6}{7}\hat{k}$$

$$\vec{F}_{BD} = F_{BD}\hat{e}_{BD} = F_{BD}\left(-\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = F_{BD}\left(-0.429\hat{i} + 0.286\hat{j} - 0.857\hat{k}\right)$$

4 pts (3 pts for unit vector and one point for magnitude of force)
Part c)

\[ \vec{R} = \vec{P} + \vec{F}_{AB} + \vec{F}_{BC} + \vec{F}_{BD} = \vec{0} \]

\[ \vec{R} = -14\hat{j} + \vec{F}_{AB}\left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right) + \vec{F}_{BC}\left(-\hat{i}\right) + \vec{F}_{BD}\left(-\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = \vec{0} \]

\[ \sum F_x = \frac{2}{3} F_{AB} - F_{BC} - \frac{3}{7} F_{BD} = 0 \quad (1) \quad 3 \text{ pts (1 point per term)} \]

\[ \sum F_y = -14 + \frac{2}{3} F_{AB} + \frac{2}{7} F_{BD} = 0 \quad (2) \quad 3 \text{ pts (1 point per term)} \]

\[ \sum F_z = \frac{1}{3} F_{AB} - \frac{6}{7} F_{BD} = 0 \quad (3) \quad 2 \text{ pts (1 point per term)} \]

Part d)

4 pts for solving the above 3 equations.
NOTE: One approach to solving the above 3 equations is shown below. There are other approaches that could be used. Therefore, students are NOT required to follow this approach.

From (3),

\[ F_{AB} = \frac{18}{7} F_{BD} = 2.571 F_{BD} \quad (4) \]

Substitute (4) into (2),

\[ -14 + \frac{2}{3}\left(\frac{18}{7} F_{BD}\right) + \frac{2}{7} F_{BD} = 0 \quad \rightarrow \quad F_{BD} = 7 \text{ kips} \]

From (4),

\[ F_{AB} = \frac{18}{7} F_{BD} = \frac{18}{7} (7) = 18 \text{ kips} \]

From (1),

\[ F_{BC} = \frac{2}{3} F_{AB} - \frac{3}{7} F_{BD} = \frac{2}{3} (18) - \frac{3}{7} (7) = 12 - 3 = 9 \text{ kips} \]

Summary

\[ F_{AB} = 18 \text{ kips} \quad \text{(compressive force in mast)} \]

\[ F_{BC} = 9 \text{ kips} \quad \text{(tensile force in cable BC)} \]

\[ F_{BD} = 7 \text{ kips} \quad \text{(tensile force in cable BD)} \]
Problem 2

Given: A force $\mathbf{F}$ of 80 lb acts along the edge DB of the tetrahedron shown in the figure below.

Find:

a) The moment of the force $\mathbf{F}$ about the point A. (14 pts)

b) The magnitude of the moment of force $\mathbf{F}$ about the axis AC. (10 pts)

c) The perpendicular distance between point A and the line of action of the force $\mathbf{F}$. (6 pts)

Solution

a)

$$\mathbf{M}_A = \mathbf{r}_{AB} \times \mathbf{F}_{DB}$$

2 pts

$$\mathbf{r}_{AB} = (20 \mathbf{j}) \text{ ft}$$

2 pts

$$\mathbf{F}_{DB} = 80 \mathbf{e}_{DB} \text{ lb} = 80 \text{ lb} \left( \frac{\mathbf{r}_{DB}}{r_{DB}} \right)$$

2 pts

$$\mathbf{r}_{DB} = (-5 \mathbf{i} + 10 \mathbf{j} - 15 \mathbf{k}) \text{ ft}$$

$$\mathbf{F}_{DB} = 80 \left( -5 \mathbf{i} + 10 \mathbf{j} - 15 \mathbf{k} \right) / (5^2 + 10^2 + 15^2)^{1/2} \text{ lb}$$

$$= (-21.38 \mathbf{i} + 42.76 \mathbf{j} - 64.14 \mathbf{k}) \text{ lb}$$

6 pts
\[ \mathbf{M}_A = (\mathbf{r}_{AB} \times \mathbf{F}_{DB}) = (20 \mathbf{j}) \text{ ft} \times (-21.38 \mathbf{i} + 42.76 \mathbf{j} - 64.14 \mathbf{k}) \text{ lb} \]
\[ = -1282.80 \mathbf{i} + 427.60 \mathbf{k} \text{ lb-ft} \]

b)

\[ M_{AC} = \mathbf{e}_{AC} \cdot \mathbf{M}_A = \mathbf{e}_{AC} \cdot (\mathbf{r}_{AB} \times \mathbf{F}_{DB}) \]
\[ \mathbf{e}_{AC} = \frac{\mathbf{r}_{AC}}{r_{AC}} \]
\[ r_{AC} = (13 \mathbf{i} + 16 \mathbf{j}) \text{ ft} \]
\[ e_{AC} = 0.63 \mathbf{i} + 0.78 \mathbf{j} \]
\[ M_{AC} = (0.63)(-1282.80) \text{ lb-ft} = -808.92 \text{ lb-ft} \]
\[ = -809 \text{ lb-ft} \]

the magnitude is positive, but the answer with negative \( M_{AC} \) is also accepted

\[ M_{AC} = 809 \text{ lb-ft} \]

c)

\[ d = \frac{M_A}{F} \]
\[ M_A = 1352.18 \text{ lb-ft} \]
\[ d = \frac{M_A}{F} = 16.9 \text{ ft} \]
Problem 3

The force, $F_B$, of 120 N is acting at vertex B of the equilateral triangle. The line of action of the force is collinear with side AB of the triangle. The length of each side of the triangle is 1 m.

a) Replace force $F_B$ by a force at point A and a couple. Express your answer in Cartesian vector form. (10 pts)

b) Replace force $F_B$ by a force at point C and a couple. Express your answer in Cartesian vector form. (10 pts)

Solution: (an alternative solution can be obtained using the cross product definition of the moment)

a) The force $F_B$ in Cartesian vector form:

$$F_B = 120\cos60^\circ i + 120\sin60^\circ j = 60i + 103.9j \text{ N}$$

5 pts

The line of action of the force $F_B$ passes through point A and thus there is no couple:

$$M_A = 0$$

5 pts

b) From the geometry, the distance between the line of action of $F_B$ and point C is equal to:

$$d_C = \sqrt{1-(1/2)^2} = \sqrt{3/4} = \sqrt{3}/2 = 0.87 \text{ m}$$

5 pts
The magnitude of the moment of the force $F_B$ about point $C$ is:

$$M_C = F_B \cdot d_C = 120 \text{ N} \times 0.87 \text{ m} = 104.4 \text{ N-m}$$

3 pts

The vector form of the moment is:

$$M_C = -104.4 \text{ N-m} \mathbf{k}$$

2 pts

where the negative sign is due to the clockwise moment that the force produces about point $C$. By the right-hand-rule, a clockwise moment acts in the negative $z$-direction.
Problem 4

Consider matrix A and B

\[ A = \begin{bmatrix} 3 & k \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \]

where \( k \) is a parameter.

a) Determine \( A A^T \), where \( T \) indicates transpose \hspace{1cm} (6 pt)

b) Determine \( A + B \) \hspace{1cm} (4pts)

c) For what values of the parameter \( k \) does \( AB = BA \)? \hspace{1cm} (10 pts)

Solution

a) 
\[
A^T = \begin{bmatrix} 3 & 0 \\ k & 3 \end{bmatrix}
\]
\hspace{1cm} 2 pts

\[
AA^T = \begin{bmatrix} 3 & k \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ k & 3 \end{bmatrix} = \begin{bmatrix} 9 + k^2 & 3k \\ 3k & 9 \end{bmatrix}
\]
\hspace{1cm} 4 pts

b) 
\[
A + B = \begin{bmatrix} 3 & k \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 + 2 & k \\ 0 & 3 + 3 \end{bmatrix} = \begin{bmatrix} 5 & k \\ 0 & 6 \end{bmatrix}
\]
\hspace{1cm} 4 pts

c) 
\[
AB = \begin{bmatrix} 3 & k \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 6 & 3k \\ 0 & 9 \end{bmatrix}
\]
\hspace{1cm} 4 pts

\[
BA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & k \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2k \\ 0 & 9 \end{bmatrix}
\]
\hspace{1cm} 4 pts

\[
AB = BA \Leftrightarrow \begin{bmatrix} 6 & 3k \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 2k \\ 0 & 9 \end{bmatrix} \Leftrightarrow k = 0
\]
\hspace{1cm} 2 pts