The Perils of Permutation

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Consider the following counting problem:

Problem 1: A group of basketball players consists of 6 members of the Lakers and 5 members of the Knicks. How many ways are there to form a 5-player team from among these 11 if at least one player must be a Knick?

The first solution usually offered by students is:

$$\binom{5}{1}\binom{10}{4}$$

We must pick one of the 5 Knicks to satisfy the requirement, they argue. We then complete the team of 5 by picking any 4 of the remaining 10 players.

Some quick calculations show that

$$\binom{5}{1}\binom{10}{4} = 5 \cdot 210 = 1050,$$

while the **total** number of ways to pick 5 players from 11 is

$$\binom{11}{5} = 462.$$

Although the argument sounds convincing, the answer is clearly too large to be correct.

Examining some examples of teams formed in this way reveals the flaw in the argument. Three such teams are illustrated below in Figure 1:

Figure 1

In each team, the first Knick listed is the one chosen to meet the requirement, while the other four players were chosen from the remaining 10.

Since order plays no role in characterizing the team, it follows that all three examples above represent the **same team**. Yet this team has been counted three times. It is this phenomenon that is responsible for the overcount.

What really went wrong? By picking a “first Knick,” this player was “distinguished,” or assigned a special role on the team and in the problem. Our solution would have been correct if the problem statement required, say, a Knick team **captain** and any four other players. In that case, the three example teams above **would** be different based on the different choices for captain. In our incorrect solution, however, we gave one player a distinguished role when none was warranted.

The correct solution is obtained by subracting the number of “Knick-free” teams from the total number, leaving the teams with at least one Knick:

$$\binom{11}{5} - \binom{6}{5} = 462 - 6 = 456.$$

The incorrect solution didn't **intentionally** impose an ordering on the players that were chosen, but distinguishing the first player creates the same effect. Consider a simpler version of the problem where only two players are chosen with the same "at least one Knick" requirement, and compare the correct and incorrect answers:

$$\text{Correct: } \binom{11}{2} - \binom{6}{2} = 55 - 15 = 40$$

$$\text{Incorrect: } \binom{5}{1}\binom{10}{1} = 50$$

In the incorrect count, the 30 teams consisting of a Knick and a Laker are counted once, but the ten teams consisting of a pair of Knicks are all double counted because distinguishing the first player has effectively imposed an unintentional ordering upon the Laker-free teams.

The example discussed above illustrates the subtlety inherent in many counting problems. Students are often surprised by the difficulty such problems cause them, presumably because counting is such a natural and familiar activity. Let's consider some other counting problems in which the unintentional imposition of ordering can lead to trouble:

Problem 2: A closet contains 10 different pairs of shoes. How many ways are there to pick six shoes from the 20 so that the 6 chosen contain **no complete pair**?

A popular incorrect solution to this problem goes as follows:

Pick the first shoe. There are 20 ways to do this. Pick the second shoe, making sure to avoid the partner of the first. There are 18 ways to do this. Proceeding this way with all six shoes, the multiplication rule yields a total of:

$$20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10$$

The problem with this solution is that it has counted **ordered** six-shoe choices. While we didn't **want** the order in which the shoes were chosen to matter, our model of the problem, which effectively envisions six ordered slots filled one at a time by the chosen shoes, **imposes** an ordering, leading again to an overcount.

In this case, recognizing the order issue allows us to correct for it. For, if there are 6! orderings of a given six shoes, then the correct count of **unordered** six-shoe choices should be:

$$\frac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10}{6!}$$

So we inadvertently introduced an ordering among the shoes and subsequently corrected for it. It would seem simpler not to introduce the ordering at all.

To avoid a "slot-based" ordering, think of the shoe-picking process as a sequence of two steps:

Step 1: Pick six of the 10 pairs, planning to use one shoe from each pair. There are $\binom{10}{6}$ ways to do this.

Step 2: From each of the six pairs, pick one of the two shoes. There are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$$

ways to do this. Thus, the total number of ways is:

$$\binom{10}{6} 2^6.$$

Note that

$$\frac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10}{6!} = \frac{2^6(10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5)}{6!} = \frac{2^6 10!}{6!4!} = \binom{10}{6} 2^6.$$

By avoiding an ordering in our model, we didn't need to worry about correcting for it.

Perhaps neither solution above seems markedly easier that the other. Sometimes, however, it is not nearly so easy to correct for the introduction of unintentional orderings.

Problem 3: How many five-card poker hands contain a "full house (three of one kind and two of another kind)" as the best hand?

Attempting to use a slot model, we argue:

Pick the first card (52 possible ways). Since the second and third cards must be of the same kind as the first, and since four of each kind exist, there are three ways to pick the second card and two ways to pick the third. This completes the "three of a kind" portion of the full house.

Now we must pick the fourth and fifth cards. The fourth card must be of a different kind than the first three, leaving 48 choices. There are then three choices for the fifth card, which has to be of the same kind as the fourth to complete the full house.

The total is thus:

$$52 \cdot 3 \cdot 2 \cdot 48 \cdot 3.$$

Unfortunately, the ordering-based error introduced in this approach is much more complicated than in the shoe problem. To investigate, let's try to solve the problem without introducing any unwanted orderings or distinguishments.

Regard the building of a full house as the following sequence of steps:

Step 1: Pick the kind to have three of. Since there are 13 kinds available, this can be done in 13 ways.

Step 2: Pick the three representatives of the kind chosen in step 1 to use in the full house. There are $\binom{4}{3}$ ways to do this.

Step 3: Pick the kind to have two of. There are 12 ways, since it must be different from the first kind chosen.

Step 4: Pick the two representatives of this kind ($\binom{4}{2}$ ways).

Thus the total number of ways is

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6 = 52 \cdot 72.$$

Remember, the slot approach yielded:

$$52 \cdot 3 \cdot 2 \cdot 48 \cdot 3 = 52 \cdot 72 \cdot 12.$$

Evidently, to correct for the unwanted ordering here, we must divide by 12, and **not** by $5! = 120$, as one might first think.

Why is this? Unlike the shoe problem, not **all** orderings of the five cards chosen have been counted. Instead, ordering has only been introduced within the "sub-hands" of sizes three and two. As a result, to correct the overcount, we must divide by $3!2! = 12$. When ordering issues become as subtle as this one, trying to correct for them is a dangerous business.

One can understand why the "slot model" is so appealing to students. After all, when six shoes, or a poker hand, or any collection of objects occurs in real life, it's always as an ordered arrangement. To avoid confusion and error, conscious attention must be paid to the following questions:

1. Is consideration of order appropriate for the counting question being asked? Sometimes, as in permutation problems, the answer is yes. In all three of the examples considered here, the answer was no.

2. If the answer above was no, can we devise a counting scheme that doesn't introduce any unwanted orderings or distinguishments? Overcounts due to uninentional orderings can escape unnoticed, and can be perilous to correct even if caught. An approach that avoids ordering completely is ideal.