MATH-2400 **Exam #1 Answers**

1. (13 pts) Solve the initial value problem:

$$x\frac{dy}{dx} - y = x^3, \quad y(1) = 0.$$

Answer:

$$y(x) = (x^3 - x)/2$$

2. (12 pts.) Find the explicit solution $y = \phi(t)$ of the initial value problem

$$\frac{dy}{dt} = \frac{1-2t}{y}, \quad y(1) = -2,$$

and find the interval where the solution is valid.

Answer:

$$y(t) = -\sqrt{2(2+t-t^2)},$$

valid for $2 + t - t^2 \geq 0$, i.e. $-1 \leq t \leq 2$.

3.(a) (9 pts) A tank initially contains 200 gallons of water and 200 grams of salt. Water containing 3 *g/gal* of salt flows into the tank at a rate of 2 *gal* per minute. The mixture in the tank also flows out of the tank at the same rate of 2 *gal/min.* Set up an initial value problem whose solution is the amount of salt in the tank at time t. **Do not solve the problem**.
Answer:

$$\frac{dS}{dt} = 6 - \frac{S}{100}, \quad S(0) = 200$$

(b) (3 pts. *BONUS*) What will be the amount of salt in the tank asymptotically as $t \to \infty$? **Answer:** $S \to 600$ *g*
(c) (3 pts.) The radioactive isotope plutonium-241 decays exponentially, $Q(t) = Q_0 e^{-0.0525t}$, where Q_0 is the initial amount and t is measured in years. Determine the half-life time of Pu-241. (It is OK to leave your answer in terms of logarithms).
Answer: The half-life time is $\ln 2/0.0525$.

4. (a) (10 pts.) Consider the equation

$$\frac{dy}{dt} = y(y-2)(5-y)$$

Find the equilibrium solutions. If the initial value is $y(0) = 1$, sketch the solution $y(t)$, find $\lim_{t\to\infty} y(t)$, the same question for $y(0) = 4$. Explain your answers.
Answer: Equilibria are $y_1 = 0, y_2 = 2, y_3 = 5$. As follows, for example, from the direction field, if $y(0) = 1$ then $y(t) \to 0$, if $y(0) = 4$ then $y(t) \to 5$.
(b) (3pts.) Given the differential equation $dy/dt = f(t, y)$. In your own words, explain what integral curves of the equation are (the geometrical meaning of solutions). **Answer:** See pp. 11-12 of the text.