Math-6500 Partial Differential Equations Fall 2007

1. PDEs of the first order

1.1. Write down an explicit solution for the initial-value problem

$$u_t + \mathbf{b}\cdot\nabla u + cu = 0, \quad x \in \mathbb{R}^n, t > 0, \quad u(x,0) = g(x).$$

Here $\mathbf{b}$ and c are constants and the gradient vector $\nabla u = (\partial u/\partial x_1, ...\partial u/\partial x_n)$ is the gradient vector. **Hint**: As an easier particular case consider the transport equation, where $n = 1$ and $c = 0$.

1.2. Solve the problem

$$xu_x + u_y = 1, \quad x \in \mathbb{R}, y > 0, \quad u(x,0) = \exp(x)$$

1.3. Use the method of characteristics to solve the problem

$$xu_x + (x+y)u_y = 1,$$

where $u(1,y) = y$ for $0 < y < 1$. Describe the region in the xy-plane over which the solution is uniquely determined.

1.4. Consider the differential equation in the *divergence form:*

$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0, \tag{div}$$

where R and S are known functions. [Note that the left-hand side is the divergence of the vector (S,R).] *Define a weak solution* $u(x,y)$ of (div) as a function for which the relation

$$\int\int (R(u)\phi_y + S(u)\phi_x)\, dx\, dy = 0 \tag{integr}$$

holds for any function $\phi(x,y)$ of class C_0^∞, i.e. for any function $\phi(x,y)$ that is infinitely differentiable and is zero everywhere except for a bounded set. (Relation (integr) follows formally from (div) through multiplication by ϕ and integration by parts.)
Let $u(x,y)$ be C^1 in each of the two regions of the xy-plane separated by the curve $x = \xi(y)$
Show that if $u(x,y)$ is a weak solution then:
(a) It is a solution of (div) in each of the two regions;
(b) It satisfies the following jump relation across the curve $x = \xi(y)$:

$$\frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)} = \frac{d\xi}{dy}$$

You may need to use the Stokes formula

$$\oint_{\partial B} Pdx + Qdy = \int\int_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

2. Properties of characteristics. General second-order equations.

2.1. Let $u(x,t)$ be a C^1-solution of $u_t + uu_x = 0$ in each of the two regions of the xy-plane separated by the curve $x = \xi(t)$. Let u be continuous but u_x have a jump discontinuity on the curve. Prove that then $d\xi/dt = u$ and hence that the curve is a characteristic.
Hint: First prove that if u is continuous across the curve then the tangential derivative of u, $du(\xi(t),t)/dx$, is also continuous. A similar argument also works for a more general equation $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$
Comment: A discontinuity in the derivative can be viewed as a disturbance in the otherwise smooth initial data; then the problem claims that the signal from this disturbance propagates along characteristics.

2.2. Let $u(x,y)$ satisfy the equation

$$u_{xx} - 2u_{xy} + u_{yy} + 3u_x - u + 1 = 0$$

in a region of the xy-plane. Classify the equation and find its characteristics. Transform it into canonical form.
Construct a solution, if possible, for each of the following Cauchy problems:
(a) $u = 2,\ u_y = 0$ on the line $y = 0$
(b) $u = 2,\ \partial u/\partial n = 0$ on the line $y + x = 0$, here $\partial u/\partial n$ is the normal derivative, i.e. the directional derivative in the direction perpendicular to the line (find its expression via u_x and u_y).
Are the data consistent with the governing equation in canonical form?

2.3. (Hodograph method or Legendre's transformation). Let $u(x,y)$ be a solution of a quasi-linear equation of the form

$$a(u_x,u_y)u_{xx} + 2b(u_x,u_y)u_{xy} + c(u_x,u_y)u_{yy} = 0.$$

Introduce new independent variables

$$\xi = u_x(x,y),\ \eta = u_y(x,y),\ \phi = xu_x + yu_y - u.$$

Prove that ϕ as a function of $\xi,\ \eta$ satisfies $x = \phi_\xi,\ \ y = \phi_\eta$ and the *linear* differential equation

$$a(\xi,\eta)\phi_{\xi\xi} - 2b(\xi,\eta)\phi_{\xi\eta} + c(\xi,\eta)\phi_{\eta\eta} = 0.$$

Comment: This trick is sometimes used to "linearize" nonlinear equations.

3. Fourier series
The problems in this section are pretty standard; they are taken from *Guenther & Lee.*
3.1. Problem 9, Sec. 3-1.
3.2. Problem 10, Sec. 3-1.
3.3. Problem 11, Sec. 3-1. Which properties of $f(x)$ make one series better than the other?
3.4. Problem 7, Sec. 3-2.

4. The wave equation in one spatial dimension

4.1. (A derivation of d'Alembert's solution). Show that the problem

$$u_{tt} - c^2 u_{xx} = 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x)$$

can be represented as the system of two transport equations

$$u_t + cu_x = w,\ w_t - cw_x = 0.$$

Find the initial conditions for the system that arise from the initial data for the wave equation; solve the system by the method of characteristics to obtain d'Alembert's formula.

4.2. (a) Solve the problem

$$u_{tt} - u_{xx} = 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x),$$

when $g(x) = 0$ and

$$f(x) = \begin{cases} x^2(1-x)^2, 0 \le x \le 1 \\ 0,\ \text{elswere} \end{cases}$$

Sketch the solution for $t = 0, 1, 2$.
(b) Repeat the problem in part (a) with different initial data: $f(x) = 0$,

$$g(x) = \begin{cases} 1 - |x|, |x| \le 1 \\ 0,\ \text{elswere} \end{cases}$$

4.3. Problem 6, Sec. 4-1 from *Guenther & Lee.* This problems explains how to split a solution of an inhomogeneous problem into a homogeneous and particular solutions (it is very similar to solving inhomogeneous ODEs). Problem 7 (not assigned!) explains, how to find a particular solution.

4.4. Problem 4, Sec. 4-2 from *Guenther & Lee (*a formal separation-of-variables Fourier series solution for the telegrapher's system). In addition, derive a second order equation for either $i(x,t)$ or $v(x,t)$.

4.5. Problem 6, Sec. 4-3 from *Guenther & Lee.* (The idea is similar to the one in Problem 4.3).

4.6. Problem 2, Sec. 4-4 from *Guenther & Lee.* Use the methods of Sec. 4-4 to solve $U_t + AU_x = 0$, where $U = (u_1, u_2)^{tr}$,

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}, U(x,0) = \begin{bmatrix} f(x) \\ 0 \end{bmatrix}, f(x) = \begin{cases} (1-x^2)^2, |x| \le 1 \\ 0, |x| > 1 \end{cases}$$

Discuss the behavior of the waves for $t > 0$.

4.7. Find a d'Alembert-type solution to the initial-boundary value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0,\ x > 0, t > 0 \\ u(x,0) &= f(x),\ u_t(x,0) = g(x),\ x \ge 0, \\ u(0,t) &= h(t),\ t \ge 0 \end{aligned}$$

by converting it to a first-order system and applying the results of Sec. 4-5 (*Guenther & Lee* Problem 5(a), Sec. 4-5).

5. Primer of distributions ($\mathcal{D}'$)

5.1. Let $f(\xi)$ be a *continuous* (not necessarily differentiable) function. Show that $u(x,t) = f(x-ct)$ is a distribution (week) solution of the partial differential equation $u_t + cu_x = 0$ (F. John, Problem 3.6.1). **Hint**: Transform the integral

$$\int\int(\phi_t + c\phi_x)u\,dx\,dt$$

to the coordinates $y_1 = x - ct,\ y_2 = t$. It is sufficient to use special test functions $\phi = \sigma(y_1)X(y_2)$.

5.2. Prove that the function $u(x_1, x_2)$ defined by

$$u(x_1,x_2) = \begin{cases} 1, & \text{for } x_1 > \xi_1, x_2 > \xi_2 \\ 0, & \text{elswere} \end{cases}$$

is a fundamental solution with the pole (ξ_1, ξ_2) of the operator $L = \partial^2/\partial x_1 \partial x_2$ in the x_1x_2-plane. (F. John, Problem 3.6.2)

5.3. Prove that the function $u(x_1, x_2)$ defined by

$$u(x_1,x_2) = \begin{cases} 1/2, & \text{for } |x_1 - \xi_1| < \xi_2 - x_2 \\ 0, & \text{elswere} \end{cases}$$

is a fundamental solution with the pole (ξ_1, ξ_2) of the operator $L = \partial^2/\partial x_2^2 - \partial^2/\partial x_1^2$ in the x_1x_2-plane. (F. John, Problem 3.6.3)

5.4. Prove that $\lim_{n\to\infty}\sin(nx) = 0$ in the space of distributions $\mathcal{D}'(\mathbb{R})$, i.e., that

$$\lim_{n\to\infty}\int \sin(nx)\phi(x)\,dx = 0$$

for any $\phi \in C_0^\infty$. Find $\lim_{n\to\infty}\sin^2(nx)$ in the space of distributions $\mathcal{D}'(\mathbb{R})$.
Comment: This example shows that multiplication of distributions is not continuous , even when it is defined.

5.5. (a) Prove that $x\delta(x) = 0$ in the sense of distributions. Thus, in $\mathcal{D}'(\mathbb{R})$ the equation $xT(x) = 0$ has a solution $T(x) = c\delta(x)$ (cf. *Guenther & Lee* Problem 10-5.4).

(b) Solve the differential equation $dT/dx = \delta(x)$ (the solution then is a fundamental solution of the equation). Don't forget to prove that your solution satisfies the equation in $\mathcal{D}'(\mathbb{R})$.

6. The Heat Equation

6.1. (Duhamel's Principle and variation of parameters). Consider the initial value problem

$$\frac{dx}{dt} + A(t)x = f(t),\ x(0) = 0,\ x \in \mathbb{R}^n$$

Let $\phi^{(1)}(t,\tau), ..., \phi^{(n)}(t,\tau)$ (each ϕ is a column vector) be a set of solutions of the homogeneous problem

$$\frac{dx}{dt} + A(t)x = 0,$$

with the initial conditions

$$\phi^{(1)}(t,\tau)|_{t=\tau} = (1,0,...,0)^{tr}, ..., \phi^{(n)}(t,\tau)|_{t=\tau} = (0,0,...,1)^{tr}$$

Thus, the matrix $\Phi(t,\tau) = [\phi^{(1)}...\phi^{(n)}]$ satisfies the (matrix) equation

$$\begin{aligned} \frac{d\Phi(t,\tau)}{dt} + A(t)\Phi(t,\tau) &= 0, \\ \Phi(t,\tau)|_{t=\tau} &= I \end{aligned}$$

(a) (Semigroup property.) Show that if $t_1 < t_2 < t_3$ then $\Phi(t_3,t_1) = \Phi(t_3,t_2)\Phi(t_2,t_1)$.
(b) Look for a solution of the inhomogeneous problem in the form

$$x = c_1(t)\phi^{(1)} + .. + c_n(t)\phi^{(n)} = \Phi c$$

Obtain a differential equation for $c(t)$ and integrate it to obtain a representation of the solution via $x(t)$ as an integral. Interpret this integral as Duhamel's integral (cf. *Guenther & Lee* Sec.5.3).

6.2. *Guenther & Lee* Problem 3, Sec. 5-3. This an analogue of the semigroup property [problem 6.1(a)] for the heat equation.

The next two problems suggest a method to solve the following problem

$$\begin{aligned} u_t &= u_{xx},\ t > 0,\ x > 0 \\ u(x,0) &= 0,\ u(0,t) = \mu(t) \end{aligned} \tag{1}$$

The result is another Duhamel's integral.

6.3. Assume that there exists a one-parametric family of functions $U(x,t;\tau)$ (τ is a parameter) that solve the heat equation

$$u_t = u_{xx},\ t > \tau,\ x > 0$$

with the initial conditions at $t = \tau$ and boundary conditions at $x = 0,\ t > \tau$

$$u(x,\tau) = 0;\ u(0,t) = 1,\ t > \tau.$$

(a) Use U to solve the problem

$$\begin{aligned} u_t &= u_{xx},\ t > t_1,\ x > 0; \\ u(x,t_1) &= 0;\ u(0,t) = 1,\ t_1 < t < t_2,\ u(0,t) = 0,\ t > t_2. \end{aligned}$$

(b) Use the result in part (a) to justify the formula for the solution of (1)

$$u(x,t) = \int_0^t \frac{\partial}{\partial\tau} U(x,t;\tau)d\tau$$

6.4. Use the similarity method to solve the problem

$$u_t = u_{xx},\ t > 0,\ x > 0;\ u(x,0) = 0;\ u(0,t) = 1,\ t > 0.$$

Namely, look for a solution $U(x,t) = v(x/\sqrt{t})$ and reduce the problem to a boundary value problem for an ordinary differential equation for $v(\xi)$, where ξ is the similarity variable. How does $U(x,t;\tau)$ from problem 6.3 relate $U(x,t)$?
6.5. Problem 5-4 #7 from *GL*.
6.6. Energy method for the heat equation. Consider the problem

$$u_t = u_{xx},\ t > 0,\ L > x > 0,\ u(x,0) = f(x) \tag{2}$$

with either $u(0,t) = u(L,t) = 0$ or $u_x(0,t) = u_x(L,t) = 0$. You may assume as much differentiability from u as you want.
(a) Show that the "energy"

$$E(t) = \int_0^L u^2(x,t)dx$$

decays as a function of t.
(b) Deduce from (a) uniqueness for the problem (2) with boundary conditions $u(0,t) = f(t)$, $u(L,t) = g(t)$.
Comment: This energy has nothing to do with any physical quantity of any significance.

7.Integral Equations

7.1. Consider the integral equation

$$u(x) = f(x) + \lambda \int_0^{10} xtu(t)dt$$

(a) For which λ does the equation have a unique solution for any f?
(b) For which λ does the equation have no solution for a generic f and infinitely many solutions for a "special" admissible f ? Describe the admissible functions f and solve the equation.
7.2. Volterra integral operators: Problem 7-4 #9(a-d) from *GL*.
7.3. Solve the integral equation

$$y(t) = t + \int_0^1 (1+st)y(s)ds$$

8. **The Laplace Equation**

1. Ill-posedness of the Cauchy problem for Laplace's equation (*cf. Guenther & Lee* Sec. 1-8). In two dimensions consider the problem

$$\Delta u = 0,\ \infty < x < -\infty,\ y > 0,$$
$$u(x,0) = f(x),\ u_y(x,0) = g(x),\ \infty < x < -\infty$$

Construct a sequence of separated solutions

$$u_n(x,y) = \frac{1}{n} Y_n(y) \cos nx$$

such that $u_n(x,0) \to 0,\ (u_n)_y(x,0) \to 0$ as $n \to \infty$, while $u_n(x,1) \to \infty$. This example shows that the solution doesn't depend continuously on the initial data: indeed, the data converge

$$F_n(x) = f(x) + u_n(x,0) \to f(x),\ G_n(x) = g(x) + (u_n)_y(x,0) \to g(x)$$

but the corresponding solutions do not converge $u(x,y) + u_n(x,y) \nrightarrow u(x,y)$.

2. (Based on Problem 4.1 #2 of *Partial Differential Equations* by F. John). Let $L = \Delta + c$ in $n = 3$ dimensions, where $c > 0$ is a constant. (L is Helmholtz or reduced wave operator).

   (a) Find all solutions of $Lu = 0$ with spherical symmetry. Hint: Set $u(r) = v(r)/r$.

   (b) Prove that

   $$\Phi(x) = -\frac{\cos(\sqrt{c}\,|x|)}{4\pi\,|x|}$$

   is a fundamental solution for L. In other words,

   1. Use the divergence theorem to prove that

   $$\int_{|x|<a} L\Phi\,dx = 0$$

   2. * Repeat the argument from *Evans* (paragraphs 2–4, pages 24-25) to demonstrate that

   $$u(x) = \int_{\mathbb{R}^3} \Phi(x-y) f(y)\,dy$$

   solves $Lu = f$.

3. (Based on Problems 8-2 #5,6 of *Guenther & Lee*).

   (a) In two dimensions use Fourier transform to solve formally

   $$\Delta u = 0,\ u(x,0) = f(x),\ \infty < x < -\infty,\ y > 0,$$

   and so find, assuming suitable behavior of u at infinity (state your assumption), that

   $$u(x,y) = \frac{y}{\pi} \int \frac{f(s)ds}{(x-s)^2 + y^2}$$

   for $y > 0$.

   (b) Now show that the formal solution of part (a) solves the problem. Namely:

   1. Check directly that the solution $u(x,y)$ satisfies $\Delta u = 0$, for $\infty < x < -\infty,\ y > 0$.

2. Prove that $u(x,y) \to f(x)$ as $y \to 0+$.

4. *Guenther & Lee* Problem 11, Sec. 8-3. The solution formula in (3-13) gives a solution of the Dirichlet problem for the Poisson equation through the dipole and volume potentials that uses Green's function G for this problem. Define a Neumann $N(x,y)$ and Robin $R(x,y)$ functions that provide similar representations for the Neumann and Robin problems.

5. (*John* 4.2 #1) Let Ω denote the domain $|x| > 1$. Let $u \in C^2(\overline{\Omega})$, $\Delta u = 0$ in Ω, *and* $\lim_{x\to\infty} u(x) = 0$. Show that
$$\max_{\overline{\Omega}} |u| = \max_{\partial\overline{\Omega}} |u|. \tag{3}$$

6. (Cf. *GL* 8-7, Problem 2) Consider the interior Robin problem with $\alpha = const$. Assume a solution in the form of a single-layer potential. Use the jump relations to find the integral equation for the density of the potential.