# Math-6500 Partial Differential Equations Fall 2007
# Assignment 5

**Due Monday, November 12, by 4pm. PDEs of the first order**

1. Write down an explicit solution for the initial-value problem

$$u_t + \mathbf{b} \cdot \nabla u + cu = 0, \quad x \in \mathbb{R}^n, t > 0, \quad u(x, 0) = g(x).$$

   Here $\mathbf{b}$ and $c$ are constants and the gradient vector $\nabla u = (\partial u/\partial x_1, ...\partial u/\partial x_n)$ is the gradient vector.

2. Solve the problem

$$xu_x + u_y = 1, \quad x \in \mathbb{R}, y > 0, \quad u(x, 0) = \exp(x)$$

3. Use the method of characteristics to solve the problem

$$xu_x + (x + y)u_y = 1,$$

   where $u(1, y) = y$ for $0 < y < 1$. Describe the region in the $xy$-plane over which the solution is uniquely determined.

4. (*F. John* 1.9.1) For the equation

$$u_x^2 + u_y^2 = u^2$$

   find

   (a) The characteristic strips;

   (b) The integral surfaces passing through the circle $x = \cos s$, $y = \sin s$, $z = 1$. (There are two surfaces).

   (c) The integral surfaces passing through the line $x = s$, $y = 0$, $z = 1$. (There are two surfaces).

5. (Characteristics of the Hamilton-Jacobi equation as extremals of a variational problem). In class I considered a one-dimensional version of the following equation

$$F := \frac{\partial u}{\partial t} + H\left(x_1, ..., x_n, t, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) = 0$$

   (Hamilton-Jacobi equation). The characteristic equations have the form

$$\frac{dx_i}{dt} = H_{p_i}, \quad \frac{du}{dt} = -H + \sum_i p_i H_{p_i}, \quad \frac{dp_i}{dt} = -H_{x_i}$$

   Setting

$$\frac{dx_i}{dt} = v_i, \quad \frac{du}{dt} = L$$

one can use the first $n+1$ equations to express $L$ as a function of $x_1, ..., x_n, t, v_1, ..., v_n$. Show then that

$$L_{v_i} = p_i, \quad L_{x_i} = -H_{x_i}.$$

This implies that

$$\frac{d}{dt} L_{v_i} - L_{x_i} = 0.$$

[These are the Euler-Lagrange equations for an extremal of the minimization problem

$$\int L\left(x_1, ..., x_n, t, \frac{dx_1}{dt}, ..., \frac{dx_n}{dt}\right) dt \to \min$$

(the least action principle); this problem is discussed in any book on analytical mechanics. See first four pages of Sec. 11.1 of *Guenther & Lee* about the Euler-Lagrange equations].

6. Consider the differential equation in the *divergence form:*

$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0, \qquad \text{(div)}$$

where $R$ and $S$ are known functions. [Note that the left-hand side is the divergence of the vector $(S, R)$.] *Define a weak solution* $u(x, y)$ of (div) as a function for which the relation

$$\int\int (R(u)\phi_y + S(u)\phi_x)\, dx\, dy = 0 \qquad \text{(integr)}$$

holds for any function $\phi(x, y)$ of class $C_0^\infty$, i.e. for any function $\phi(x, y)$ that is infinitely differentiable and is zero everywhere except for a bounded set. (Relation (integr) follows formally from (div) through multiplication by $\phi$ and integration by parts.). Let $u(x, y)$ be $C^1$ in each of the two regions of the $xy$-plane separated by the curve $x = \xi(y)$. Show that if $u(x, y)$ is a weak solution then:

(a) It is a solution of (div) in each of the two regions;

(b) It satisfies the following jump relation across the curve $x = \xi(y)$ :

$$\frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)} = \frac{d\xi}{dy}$$

You may need to use the Stokes formula

$$\oint_{\partial B} P dx + Q dy = \int\int_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$