Math-6500 Partial Differential Equations Fall 2007
Assignment 4

Due Thursday, October 25, by 4pm.
Hermite Functions and the Fourier Transform (These problems are optional)

1. Define

$$(a^\dagger f)(x) := \frac{1}{\sqrt{2}}[xf(x) - \frac{d}{dx}f(x)]$$
$$(af)(x) := \frac{1}{\sqrt{2}}[xf(x) + \frac{d}{dx}f(x)]$$

It is easy to see that

$$aa^\dagger - a^\dagger a = 1 \tag{1}$$

In the space $L^2(-\infty, \infty)$ consider

$$\Phi_0(x) = \exp(-\frac{x^2}{2}), \Phi_1(x) = (a^\dagger \Phi_0)(x), ..., \Phi_n = (a^\dagger)^n \Phi_0, ...$$

(a) Prove that vectors Φ_j are orthogonal and determine their norms. Hint: Use the identity

$$\left\langle (a^\dagger)^n \Phi_0, (a^\dagger)^m \Phi_0 \right\rangle = \left\langle a(a^\dagger)^n \Phi_0, (a^\dagger)^{m-1} \Phi_0 \right\rangle$$

and the commutation relation and argue by induction.

(b) Prove that $\{\Phi_j\}$ form a basis in $L^2(-\infty, \infty)$.
Comment: The normalized functions $h_n(x) = \Phi_n(x)/\left\|\Phi_n\right\|$ are called Hermite functions and the corresponding polynomial factors

$$H_n(x) = \exp(\frac{x^2}{2}) h_n(x)$$

are Hermite polynomials. Obviously they are orthonormal with the weight $\exp(-x^2)$:

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = \delta_{mn}$$

(c) Hermite functions are eigenfunctions of the Fourier transform. Find the eigenvalues, and from this deduce that the Fourier transform preserves the inner product: $\langle f, g \rangle = \langle \mathcal{F}f, \mathcal{F}g \rangle$, where $\mathcal{F}$ is the Fourier transform in $L^2(-\infty, \infty)$.
Comment: This result is called the Plancherel theorem. See any textbook for a more traditional proof.

The wave equation in one spatial dimension

1. (a) Solve the problem

$$u_{tt} - u_{xx} = 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x),$$

when $g(x) = 0$ and

$$f(x) = \begin{cases} x^2(1-x)^2, 0 \le x \le 1 \\ 0,\ \text{elsewhere} \end{cases}$$

Sketch the solution for $t = 0, 1, 2$.

(b) Repeat the problem in part (a) with different initial data: $f(x) = 0$,

$$g(x) = \begin{cases} 1 - |x|, |x| \le 1 \\ 0,\ \text{elsewhere} \end{cases}$$

2. Problem 6, Sec. 4-1 from *Guenther & Lee.* This problem explains how to split a solution of an inhomogeneous problem into a homogeneous and particular solutions (it is very similar to solving inhomogeneous ODEs). Problem 7 explains, how to find a particular solution.

3. Problem 7, Sec. 4-1 from *Guenther & Lee.* This problem provides very detailed instructions on how to obtain a Duhamel formula for the solution of

$$u_{tt} - u_{xx} = F(x,t),\ u(x,0) = 0,\ u_t(x,0) = 0.$$

Comment: you don't need to understand, why the characteristic variables are given this name.

4. Problem 4, Sec. 4-2 from *Guenther & Lee (*a formal separation-of-variables Fourier series solution for the telegrapher's system). In addition, derive a second order equation for either $i(x,t)$ or $v(x,t)$.

5. Problem 6, Sec. 4-3 from *Guenther & Lee.* (The idea is similar to the one in Problem 6 from 4.1 above).