Math-6500 Partial Differential Equations Fall 2007
Assignment 3

Due Thursday, October 11, by 4pm.

Elementary Sobolev spaces.

1. (Best constant in Poincaré's inequality). Poincaré's inequality states that if $u \in \overset{\circ}{H}_1(\Omega)$ then
$$\int_\Omega u^2 dx \le \frac{1}{C}\int_\Omega |Du|^2\, dx.$$
From here it is clear that for the universal constant C one gets
$$C \le \frac{\int_\Omega |Du|^2\, dx}{\int_\Omega u^2 dx}.$$
Show that if there exists a function $u \in C^2(\bar{\Omega})$ vanishing on $\partial\Omega$ for which the quotient above attains its least value λ, then $\Delta u + \lambda u = 0$. I.e., u is an eigenfunction of the operator $-\Delta$ with zero boundary conditions. In fact, λ must be the smallest eigenvalue.

   Hint: Consider the problem of minimization of the Dirichlet integral
$$\int_\Omega |Du|^2\, dx \to \min$$
under the constraint
$$\int_\Omega u^2 dx = 1.$$

2. Prove that $\phi(x) \equiv 1$ cannot be approximated in the $H_1(0,1)$ norm by functions from $\overset{\circ}{H}_1(0,1)$. **Hint:** A possible approach to this problem is to show, by using Poincaré's inequality, that for any $u \in \overset{\circ}{H}_1(0,1)$
$$\|\phi - u\|_{H_1} \ge c$$

The heat equation

3. Prove directly that the function
$$\Phi(x,t) = \frac{1}{(4\pi t)^{1/2}}\exp\{-\frac{x^2}{4t}\}H(t), \quad H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$
($H(t)$ is the Heaviside function), solves the equation $(\partial_t - \Delta)\,\Phi = \delta(x)\delta(t)$ in the sense of distributions. I.e., for any $\in C_0^\infty$
$$-\int\int (\phi_t + \phi_{xx})\,\Phi(x,t)dx\,dt = \phi(0,0)$$
Hint: Remove a neighborhood of the singularity $S(\varepsilon) = (-\varepsilon,\varepsilon)\times(0,\varepsilon)$ and by using Green's formula show that
$$I_\varepsilon = -\int\int_{R^2 - S(\varepsilon)} (\phi_t + \phi_{xx})\,\Phi(x,t)dx\,dt \to \phi(0,0)$$

4. Energy method for the heat equation. Consider the problem

$$u_t = u_{xx},\ t > 0,\ L > x > 0,\ u(x,0) = f(x) \tag{1}$$

with either $u(0,t) = u(L,t) = 0$ or $u_x(0,t) = u_x(L,t) = 0$. You may assume as much differentiability from u as you want.

   (a) Show that the "energy"

   $$E(t) = \int_0^L u^2(x,t)dx$$

   decays as a function of t.

   (b) Deduce from (a) uniqueness for the problem (1) with boundary conditions $u(0,t) = f(t)$, $u(L,t) = g(t)$. **Comment:** This energy has nothing to do with any physical quantity of significance.

5. (Duhamel's Principle and variation of parameters). Duhamel's Principle provides a method of constructing a solution of the evolutionary problem with the inhomogeneity by integrating (with respect to τ) a family of homogeneous solutions $\phi(t,\tau)$ with initial conditions at $t = \tau$.Consider the initial value problem

$$\frac{dx}{dt} + A(t)x = f(t),\ x(0) = 0,\ x \in \mathbb{R}^n$$

Let $\phi^{(1)}(t,\tau), ..., \phi^{(n)}(t,\tau)$ (each ϕ is a column vector) be a set of solutions of the homogeneous problem

$$\frac{dx}{dt} + A(t)x = 0,$$

with the initial conditions

$$\phi^{(1)}(t,\tau)|_{t=\tau} = (1,0,...,0)^T, ..., \phi^{(n)}(t,\tau)|_{t=\tau} = (0,0,...,1)^T$$

Thus, the matrix $\Phi(t,\tau) = [\phi^{(1)}...\phi^{(n)}]$ satisfies the (matrix) equation

$$\frac{d\Phi(t,\tau)}{dt} + A(t)\Phi(t,\tau) = 0,$$

$$\Phi(t,\tau)|_{t=\tau} = I$$

   (a) (Semigroup property.) Show that if $t_1 < t_2 < t_3$ then $\Phi(t_3,t_1) = \Phi(t_3,t_2)\Phi(t_2,t_1)$.

   (b) Look for a solution of the inhomogeneous problem in the form

   $$x = c_1(t)\phi^{(1)} + .. + c_n(t)\phi^{(n)} = \Phi c$$

   Obtain a differential equation for $c(t)$ and integrate it to obtain a representation of the solution via $x(t)$ as an integral. Interpret this integral as Duhamel's integral (cf. *Guenther & Lee* Sec. 5.3).

6. Use the similarity method to solve the problem

$$u_t = u_{xx},\ t > 0,\ x > 0;\ u(x,0) = 0;\ u(0,t) = 1,\ t > 0.$$

Namely, look for a solution $U(x,t) = v(x/\sqrt{t})$ and reduce the problem to a boundary value problem for an ordinary differential equation for $v(\xi)$, where ξ is the similarity variable.

7. Assume that there exists a one-parametric family of functions $U(x,t;\tau)$ (τ is a parameter) that solve the heat equation

$$u_t = u_{xx},\ t > \tau,\ x > 0$$

with the initial conditions at $t = \tau$ and boundary conditions at $x = 0,\ t > \tau$

$$u(x,\tau) = 0;\ u(0,t) = 1,\ t > \tau.$$

(a) Combine solutions of the family $U(\tau)$ to solve the problem

$$u_t = u_{xx},\ t > t_1,\ x > 0;$$
$$u(x,t_1) = 0;\ u(0,t) = 1,\ t_1 < t < t_2,\ u(0,t) = 0,\ t > t_2.$$

(b) Use the result in part (a) to justify the following Duhamel's formula

$$u(x,t) = \int_0^t \mu(\tau)\frac{\partial}{\partial\tau}U(x,t;\tau)d\tau$$

for the solution of the problem

$$u_t = u_{xx},\ t > 0,\ x > 0$$
$$u(x,0) = 0,\ u(0,t) = \mu(t) \qquad (2)$$

Comment: Of course, $U(x,t)$ of problem 6 is $U(x,t;\tau = 0)$ of the current problem.