Math-6500 Partial Differential Equations Fall 2007
Assignment 2

Due Thursday, September 27, by 4pm.

1. One more problem on distributions (that I mentioned in class several times, but never assigned). Let $f \in L^1(-\infty, \infty)$ with $\int f dx = 1$. Prove that $\lim_{n\to\infty} nf(nx) = \delta(x)$ in the space of distributions $\mathcal{D}'(\mathbb{R})$, i.e., that

$$\lim_{n\to\infty} \int nf(nx)\phi(x)\,dx = \phi(0)$$

for any $\phi \in \mathcal{D} = C_0^\infty$. Actually this holds for any bounded, continuous ϕ. If you are not sure what L^1 is you can take a continuous function *such that* $\int |f|\,dx$ *converges*; you can even limit yourself to a continuous finite function (nonzero on a finite interval).

2. Use the method of images to construct Green's function for the quarter-plane $\{x_1 > 0,\ x_2 > 0\}$ (mimic the derivation of the Green's function for the half-space). Sketch a proof of convergence of the solution to the desired boundary values.

3. (**optional, part (a) was solved in class**) Consider the Dirichlet problem on the unit disk. Let $f(\theta)$ denote the boundary value function, assumed to be continuous. Let $u(r\cos\theta, r\sin\theta)$ be a solution of $\Delta u = 0$, $u = f$ on $r = 1$.

   (a) Show that for $r < 1$

$$u = \sum_{k=0}^{\infty} (a_k \cos\theta + b_k \sin\theta) r^k$$

   where a_k, b_k are the Fourier coefficients of $f(\theta)$. Hint: Use separation of variables to show that for $r < 1$, u has an expansion with certain coefficients; then from the boundary conditions see that the coefficients must be the Fourier coefficients of f.

   (b) Show that the Dirichlet integral of u is given by

$$\|u\|^2 = \pi \sum_{k=0}^{\infty} k(a_k^2 + b_k^2)$$

   (c) Find sequences a_k, b_k so that the series above diverges, while the series

$$\sum_{k=0}^{\infty} (|a_k| + |b_k|) < \infty$$

   The latter guarantees f being continuous, while the divergence of $\|u\|^2$ means that a solution to the Dirichlet problem with f as a boundary condition cannot be found from the Dirichlet principle.

   Comment: This example belongs to Weierstrass. It effectively burried the Dirichlet principle until Hilbert realized how to fix it.

Strong and weak derivatives. These exercises are based on Section 11.4 and problems 3-4, 8-10, 13-14 of *Guenther & Lee.* You may use results of problems 11-12 on mollifiers, which are explained in detail in Section C.4 of *Evans.* The goal of the exercises is to clarify the notion of $H_1(D)$. For simplicity take D to be a 1D interval $D = (a, b)$, $\bar{D} = [a, b]$

4. Strong derivative. Let $f \in H_1$. Suppose that $\{f_n\} \subset C^1(\bar{D})$ and $f_n \to f$ in H_1. It is proved in the text that then df_n/dx converges in $L_2(D)$ to some function g, which is called the *strong derivative* of f. It can be shown that g does not depend on the choice of a particular sequence $\{f_n\}$ (problem 11-4 #3). If $f \in C^1(\bar{D})$ then it also belongs to $H_1(D)$. Thus, in addition to its classical ("normal") derivative df/dx it also has the strong derivative, g as defined above. **Demonstrate** that $g = df/dx$ (this problem is quite easy). We will keep the notation df/dx for the strong derivative g, even when the derivative is not classical.

5. Weak derivative. Let $f \in L_2(D)$. We call h the *weak derivative* of f if for any $\zeta \in C_0^1(\bar{D})$

$$\left\langle f, \frac{d\zeta}{dx} \right\rangle = -\langle h, \zeta \rangle, \text{ where } \langle \phi, \psi \rangle = \int_D \phi\psi dx$$

This is similar to the distributional derivative, just the class of test functions is broader, $C_0^1(\bar{D}) \supset \mathcal{D} = C_0^\infty(\bar{D})$. **Show** that any $f \in H_1(D)$ possesses the weak derivative h which coincides with its strong derivative, $h = df/dx$.

6. This problem shows that if u is weakly differentiable then it is strongly differentiable. Let $u \in L_2(D)$ and u vanishes off some bounded closed $K \subset D$. The mollifications u_ε of u converge to u in $L_2(D)$ (problem 11-4 #12). Suppose that u has a weak derivative h. **Prove** that $u'_\varepsilon \to h$, thus h is a strong derivative of u.