

# Dynamics of Thermally-Insulated Nonequilibrium Stefan Problem

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## Abstract

We study a two-phase Stefan problem with kinetics. Our results obtained earlier [15] rely on an artificial damping represented by the heat losses. The evidence from DNS, however, suggests that this additional damping is unnecessary. Here we prove existence of a finite-dimensional attractor for the problem without heat losses. We use a more elegant technique of energetic type estimates in appropriately defined weighted Sobolev spaces as opposed to the parabolic potentials of [15]. We demonstrate existence of compact attractors in the Sobolev spaces and prove that the attractor consists of sufficiently regular functions. This allows us to show that the Hausdorff dimension of the attractor is finite.

**Key words:** free interface, compact attractor, Hausdorff dimension  
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**Running head:** Thermally-Insulated Nonequilibrium Stefan Problem

## 1 Introduction

The principal result of the present paper is the proof of existence of a finite-dimensional compact attractor for the two-phase Stefan problem with kinetic condition *without heat losses*. This Free-Interface Problem (FIP) is employed to model propagation of condensed phase combustion fronts and some phase transition interfaces (see, for example, [25, 17, 27]). The condensed phase combustion, also known as Combustion Synthesis, finds technological applications in synthesis of some technologically advanced materials, see [21], [28] and also [29] for a popular exposition.

The FIP with kinetics is known to generate a variety of complex thermokinetic oscillations [9] such as Hopf bifurcation, period doubling cascades resulting in chaotic dynamics, a Shilnikov-Hopf bifurcation etc. In the appropriately

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nondimensionalized variables the problem is formulated as follows: find  $s(t)$  and  $u(x, t)$  such that

$$u_t = u_{xx}, \quad x \neq s(t), \quad t > 0, \quad (1.1)$$

$$u(x, 0) = u_0(x) \geq 0, \quad (1.2)$$

$$g[u(s(t), t)] = v(t) < 0, \quad (1.3)$$

$$[u_x(s(t), t)] := u_x^+(s(t), t) - u_x^-(s(t), t) = v(t) \quad (1.4)$$

where  $v(t)$  is the interface velocity,  $s(t) = \int_0^t v(\tau) d\tau$  is its position,  $u$  is the temperature, and the one-sided derivatives  $u_x^+$  and  $u_x^-$  are taken at the free interface. We would like to stress that, in contrast with our previous work [15], in the heat equation (1.1) the Newton's cooling term is absent.

The evolution of the model in (1.1)-(1.4) is governed by the interplay between the heat release due to the kinetics  $g(u|_{x=s(t)})$  and the heat dissipation by the medium. The jump condition (1.4) (the Stefan boundary condition) reflects the balance between the heat production at the free boundary and its diffusion by the medium. The condition (1.3) is a manifestation of the *nonequilibrium* nature of the transition; for the classical (equilibrium) Stefan problem it is replaced by the condition  $u|_{x=s(t)} = 0$ . We also require that the solution at minus infinity (far ahead of the advancing interface) should satisfy  $\lim_{x \rightarrow -\infty} u(x, t) = 0$ . This requirement is related to the front propagation in the negative  $x$ -direction ( $v < 0$ ).

Numerical calculation of the correlation dimension based on direct numerical simulation of FIP (1.1)-(1.4) suggests that the essential dynamics of the free-interface problem should be finite-dimensional. Indeed, we were able to prove in [15] that a variant of FIP (1.1)-(1.4) possesses a compact finite-dimensional attractor.

However, the crucial component used in the proofs of the results presented in [15] was the addition of a heat loss term to the heat equation. Thus, according to [15] the dissipative character of the problem, compactness of the attractor and the very estimate of its Hausdorff dimension seem in a sense to be due to this additional artificial damping. Meanwhile our numerical experience with FIP (1.1)-(1.4) clearly suggested that should not be the case, although we were not able to prove it at the time.

The analysis presented in [15] was carried out in weighted spaces of continuous functions first, and only after that in the (weighted) Hilbert spaces that are necessary for the estimates of the Hausdorff dimension. Thus, the results of [15] were not quite satisfactory as they were based on artificial damping, and additionally the analysis was unnecessarily cumbersome due to purely technical difficulties.

*In the present paper we manage to remove the requirement of nonzero heat losses.* Using weighted Sobolev spaces directly, with somewhat different weights, one can obtain compactness and finite dimensionality results without the heat loss term. Also we use a more elegant technique of energy type estimates in appropriately defined weighted Sobolev spaces in contrast with the parabolic potentials of [15].

Next, for reader's convenience we present a detailed outline of the logical structure of the paper. In Sec. 2 we collect some pertinent background information. In Sec. 3 we state the main existence results of the paper (Theorem 3) in weighted Sobolev spaces  $H_\omega^1$  with initial data in  $H_\omega$  (see (3.7) for the definition). The proof of the theorem is quite lengthy and consists of several logical steps that are carried out in the Sections 4-6. First in Sec.4 we prove existence of the interface velocity, and derive a global estimate for the interface temperature. Once the interface velocity is determined, one can construct a classical solution explicitly via the representation formula (4.12) below. The a priori energy estimates (5.22) in Sections 5-6 then demonstrate that the solution belongs to appropriate functional spaces and is bounded as stated in Theorem 3. This implies global existence. Even more importantly (5.22) demonstrates the existence of an absorbing set for the free-interface problem and therefore establishes dissipativity of the problem.

There is a substantial literature that treats basic existence and well-posedness issues for initial-boundary value problems for different sharp-interface models with kinetics, related to the problem (1.1)-(1.4), see, for example, [19, 22, 3, 5]. These works are mostly concerned with local in time existence. In recent papers Brauner *et al.* and Lorenzi, [1, 2, 18] study weakly-nonlinear dynamics for related problems. In particular they consider perturbations of traveling-wave initial data and investigate their instability and bifurcations. In contrast, the present paper focuses on *strongly nonlinear dynamics for a wide range of initial data and parametric regimes*.

In subsequent sections we obtain estimates, which allow us to prove compactness of the attractor later on. The solution of (1.1)-(1.4) can naturally be viewed as a sum of two parts: a contribution from the interface and such from the initial data. The estimate (5.27) shows exponential decay for the  $H_\omega$ -norm of the contribution from the initial data, while (5.30) provides a uniform bound for the contribution from the interface. The additional ingredient in the proof of compactness of the attractor is presented in Sec. 5.2: the norms of the spatial "tails" of the interface contributions decay uniformly in time, (5.31)-(5.32).

In Sec. 6 we obtain uniform bounds on the  $H_\omega$ -norm of the derivative (6.35). The bounds are derived through the potential theory representation, however we remark that one could also obtain them via energy estimates, combined with the Uniform Gronwall Lemma [26, Chap. 3]. The estimate in (6.35) is used in the compactness proof, as well as in the dimension estimate in Sections 9 and 10.

In Sec. 7 we prove that solutions of the free boundary problem (1.1)-(1.4) depend continuously on initial data (Theorem 18); in other words we establish continuity of the corresponding evolution operator. This fact is a necessary prerequisite for the general theorems used in Sec. 8 to hold. Another application of the well-posedness Theorem 18 is for the proof of differentiability of the evolution operator in Sec. 10.

In Sec. 9 we estimate the Hausdorff dimension of the attractor based on the techniques described for instance in [26]. We study evolution of the infinitesimal volume along the trajectories in the attractor and demonstrate that for

sufficiently large  $m$  that is defined solely by the physical properties of the problem, the  $m$ -dimensional volume decays exponentially. After that in Sec. 10 we prove that the evolution operator is uniformly differentiable with respect to the initial data. This, combined with the estimate for the linearized evolution of the infinitesimal volume, leads to the conclusion that the Hausdorff dimension of the attractor is finite.

We conclude with a remark on the notation convention: unless otherwise indicated, *all the spatial integrals are assumed to be on the whole real line*, i.e., we always write  $\int \phi(x)dx$  instead of  $\int_{-\infty}^{\infty} \phi(x)dx$ .

## 2 Properties of solutions: Previous results

In this section we present some pertinent background information from [8] (certain statements are slightly modified and clarified). The following theorem summarizes existence results:

**Theorem 1** *Suppose that the kinetic functions  $g$  satisfies the following assumptions:*

(A1)  $g(u)$  is a continuously differentiable, negative function, with  $g'(u) < 0$  on  $[0, \infty)$  and  $g(0) = -v_0$  for some velocity  $-v_0 < 0$

(A2)  $g(u)$  is sublinear:  $\lim_{u \rightarrow \infty} g(u)/u = 0$ .

*Let the initial data  $u_0(x) \in C(-\infty, \infty)$ . Then there exists one and only one classical solution of the free interface problem (1.1)-(1.4). The solution is uniformly bounded for all  $t > 0$ .*

For the clarity of presentation, on many occasions in the sequel we replace the sublinearity condition (A2) by a stronger condition: We will assume that  $g(u)$  is a monotonically decreasing differentiable function on  $[0, \infty]$  with  $|g'| \leq L$  and satisfying

$$-V_0 \leq g(u) \leq -v_0 \text{ for some } V_0, v_0 > 0. \quad (2.5)$$

These conditions are satisfied, for instance, for the standard Arrhenius kinetics, (in the context of condensed state combustion see, for example, [9]). *However all the results of the paper hold for the less restrictive sublinear kinetics as well.*

Under the assumptions of Theorem 1 and some additional conditions the following smoothness result holds [8]:

**Theorem 2** *Let the initial data  $u_0$  be twice differentiable in  $x < 0$  and  $x > 0$  with bounded derivatives. Let  $u_0$  satisfy the following matching condition:*

$$g(u_0(0)) = \frac{\partial u_0^+}{\partial x}(0) - \frac{\partial u_0^-}{\partial x}(0), \quad (2.6)$$

*in addition, let the derivative of the kinetics function  $g'$  be Lipschitz continuous. Then the velocity  $v$  is differentiable.*

### 3 Statement of the existence result

For our purposes we need to study properties of solutions in appropriate weighted spaces that are introduced next. Let  $\omega(x) := e^{-\alpha x}$ ,  $\alpha > 0$ ; we define the Hilbert spaces

$$\begin{aligned} H_\omega &= \{f \mid \|f\|_\omega := (\int \omega(x) |f|^2 dx)^{1/2} < \infty\} \\ H_\omega^1 &= \{f \mid \|f\|_{\omega,1} := \|f\|_\omega + \|\partial_x f\|_\omega < \infty\} \end{aligned} \quad (3.7)$$

The choice of the weight  $\omega$  is dictated by the spectral properties of the linearization of (1.1)-(1.4): an appropriate choice of the parameter  $\alpha$  allows one to shift into the negative real part half-plane the essential spectrum of the linearization about any solution from the attractor. It was shown in [23] that the continuous spectrum of the linearization about the basic traveling wave solution is contained within a parabola in the negative half-plane  $Re z < -c(Im z)^2$ , for some constant  $c > 0$ , while the origin  $z = 0$  is an accumulation point for the spectrum. Following ideas of Sattinger [24], introduction of the weighted spaces allows us to move the continuous spectrum away from zero, which is instrumental in the proof of a version of the Hopf theorem. Details concerning a rigorous treatment of the Hopf bifurcation for the sharp interface models can be found in [23], [7].

To follow the evolution in weighted spaces we recast the problem in the interface-attached coordinate frame. We introduce  $\zeta = x - s(t)$ , then the dependent variable becomes  $\tilde{u}(\zeta, t) = u(\zeta + s(t), t)$ ; with the customary abuse of notation we still use  $u(\zeta, t)$  for  $\tilde{u}(\zeta, t)$ . Then the problem takes the form:

$$u_t = u_{\zeta\zeta} + v(t)u_\zeta, \quad \zeta \neq 0, \quad t > 0, \quad (3.8)$$

$$g[u(0, t)] = v(t), \quad (3.9)$$

$$[u_\zeta(0, t)] := u_\zeta^+(0, t) - u_\zeta^-(0, t) = v(t), \quad (3.10)$$

$$u(\zeta, 0) = u_0(\zeta) \geq 0,$$

All the norm estimates in the subsequent sections should be understood in the  $\zeta t$ -coordinates. Alternatively the estimates can be carried out in the original coordinates, with the shifted weight function  $\omega(x - s(t))$ . Thus, everywhere for any function  $F$

$$\|F(\cdot, t)\|_\omega := (\int |F(\zeta, t)|^2 \omega(\zeta) d\zeta)^{1/2} = (\int |F(x, t)|^2 \omega(x - s(t)) dx)^{1/2}$$

(Here and below we use a convenient notation " $:=$ " which always means "equals by definition").

The main existence result of the paper is the following global existence theorem for the free interface problem *with initial data in  $H_\omega$*  (cf. Theorem 1):

**Theorem 3** *Suppose that the kinetic function  $g(u)$  is a continuously differentiable, monotone decreasing, negative function on  $[0, \infty)$  with  $-V_0 \leq g(u) \leq -v_0$*

for some  $V_0, v_0 > 0$ , and  $|g'| \leq L$  for some  $L > 0$ . Suppose the initial data  $u_0 \in H_\omega$ . Then

(i) There exists one and only one solution of the free interface problem (1.1)-(1.4) such that  $u \in C([0, \infty), H_\omega) \cap C((0, \infty), H_\omega^1)$  and  $v \in C(0, \infty)$ .

(ii) Moreover,  $\|u(\cdot, t)\|_\omega \leq C$  for all  $t$ , while  $\|u(\cdot, t)\|_{\omega,1} \leq C_1$  for  $t \geq t_0 > 0$ , where  $C_1$  depends on  $t_0$ .

(iii) The solution is classical and uniformly bounded for all  $t \geq t_0 > 0$ .

The proof of the theorem is quite lengthy and consists of several logical steps that are carried out in the Sections 4-6. First in Sec.4 we prove existence of the interface velocity, and derive a global estimate for the interface temperature. Once the interface velocity is determined, one can construct a classical solution explicitly via the representation formula (4.12) below. The a priori energy estimates (5.22) in Sections 5-6 then demonstrate that the solution belongs to appropriate functional spaces and is bounded as stated in Theorem 3. This implies global existence.

## 4 Integral equation for the interface velocity

### 4.1 Integral representation for solutions

The proof of Theorem 3 utilizes the reduction to an integral equation for the interface velocity,

$$v(t) = g \left( \int G(s(t) - \xi, t) u_0(\xi) d\xi - \int_0^t G(s(t) - s(\tau), t - \tau) v(\tau) d\tau \right). \quad (4.11)$$

The latter arises from the interface boundary condition, when the solution is sought via the single layer potential representation (see [8]):

$$\begin{aligned} (Tu_0)(\zeta, t) &: = u(\zeta, t) = - \int_0^t G(\zeta + s(t) - s(\tau), t - \tau) v(\tau) d\tau \\ &+ \int G(\zeta + s(t) - \xi, t) u_0(\xi) d\xi, \end{aligned} \quad (4.12)$$

where

$$G(x - \xi, t - \tau) = \exp\left\{-\frac{(x - \xi)^2}{4(t - \tau)}\right\} [4\pi(t - \tau)]^{-1/2}$$

is the heat kernel and  $s(t) = \int_0^t v(\tau) d\tau$ .

It is convenient to view the representation formula (4.12) for the solution operator  $T$  as a sum of the two parts: the contribution of the free boundary

$$u_1(\zeta, t) := T_1(t)u_0(\zeta) := - \int_0^t G(\zeta + s(t) - s(\tau), t - \tau) v(\tau) d\tau \quad (4.13)$$

and that of the initial data

$$u_2(\zeta, t) := T_2(t)u_0(\zeta) := \int G(\zeta + s(t) - \xi, t)u_0(\xi)d\xi \quad (4.14)$$

Next we prove local existence for the integral equation (4.11).

## 4.2 Local existence for interface velocity

Let  $Kv$  denote the action of the right hand side of (4.11) with a fixed  $u_0 \in H_\omega$ . Our first step is to show that  $K$  is a contraction.

**Proposition 4** *The transformation  $\psi = K\varphi$  is a contraction on a closed subset  $E_\sigma$  of  $C(0, \sigma]$  for some  $\sigma$ ,  $E_\sigma := \{v : v_0 \leq -v(t) \leq V_0, 0 < t \leq \sigma\}$  for some  $\sigma > 0$ . Therefore  $K$  has a unique fixed point  $v = Kv$ .*

**Proof.** Introduce

$$|\phi|_\sigma := \sup_{0 < t \leq \sigma} |\phi(t)|$$

First we note that  $K$  maps  $E_\sigma$  into itself, since the range of the function  $g$  is  $[-V_0, -v_0]$ . Now, let  $\psi = K\varphi$  and  $\psi' = K\varphi'$ , then

$$\begin{aligned} |\psi - \psi'| &\leq L \left| \int [G(s - \xi, t) - G(s' - \xi, t)]u_0(\xi)d\xi \right. \\ &\quad \left. + \int_0^t [G(s(t) - s(\tau), t - \tau)\varphi - G(s'(t) - s'(\tau), t - \tau)\varphi']d\tau \right| \\ &= L \left| \int [G(s - \xi, t) - G(s' - \xi, t)]u_0(\xi)d\xi + \int_0^t \Delta G\varphi'(\tau)d\tau \right. \\ &\quad \left. + \int_0^t G(s(t) - s(\tau), t - \tau) (\varphi - \varphi') d\tau \right| := L |W_1 + W_2 + W_3| \end{aligned} \quad (4.15)$$

where  $L$  is the Lipschitz constant for  $g$ ,  $s(t) = \int_0^t \varphi(\tau)d\tau$ ,  $s'(t) = \int_0^t \varphi'(\tau)d\tau$ , and

$$|\Delta G| := |G(s(t) - s(\tau), t - \tau) - G(s'(t) - s'(\tau), t - \tau)|$$

The estimation of  $|\Delta G|$  is quite elementary and is based on the mean value theorem. First we note that

$$\begin{aligned} |\Delta G| &= |s(t) - s(\tau) - (s'(t) - s'(\tau))| \left| \frac{\partial G}{\partial x}(\tilde{s}, t - \tau) \right| \\ &= \left| \frac{s(t) - s'(\tau) - (s(\tau) - s'(\tau))}{2(t - \tau)} \right| |\tilde{s}G(\tilde{s}, t - \tau)| \\ &= \frac{1}{2} \left| \frac{ds}{dt}(\tau') - \frac{ds'}{dt}(\tau') \right| |\tilde{s}G(\tilde{s}, t - \tau)| = \frac{1}{2} |\varphi(\tau') - \varphi'(\tau')| |\tilde{s}G(\tilde{s}, t - \tau)| \end{aligned}$$

where  $\tau \leq \tau' \leq t$  and  $\tilde{s}$  is between  $s'(t) - s'(\tau)$  and  $s(t) - s(\tau)$ . Since

$$|\tilde{s}| \leq \max \{|s'(t) - s'(\tau)|, |s(t) - s(\tau)|\} \leq V_0(t - \tau)$$

and  $|G| \leq C_0(t - \tau)^{-1/2}$  we get the estimate

$$|\Delta G| \leq C_0 V_0 |\varphi(\tau') - \varphi'(\tau')| (t - \tau)^{1/2}. \quad (4.16)$$

Integration with respect to  $\tau$  yields:

$$\begin{aligned} |W_2 + W_3| &\leq \left| \int_0^t \Delta G \varphi'(\tau) d\tau \right| + |\varphi - \varphi'|_\sigma \int_0^t C_0(t - \tau)^{-1/2} d\tau \\ &\leq \left( \frac{2}{3} V_0 C_0 t^{3/2} + \frac{C_0}{2} t^{1/2} \right) |\varphi - \varphi'|_\sigma := \left( A_2 t^{3/2} + A_3 t^{1/2} \right) |\varphi - \varphi'|_\sigma \end{aligned} \quad (4.17)$$

The estimation for the term  $|W_1|$  in (4.15) is a little more involved. Denote

$$\delta G := G(s - \xi, t) - G(s' - \xi, t).$$

Then

$$|W_1| = \left| \int \delta G u_0(\xi) d\xi \right| = \left| \int \omega \omega^{-1} \delta G u_0 d\xi \right| \underset{\text{Cauchy-Schwarz}}{\leq} \sqrt{I_1} \|u_0\|_\omega \quad (4.18)$$

where

$$I_1 = \int |G(s - \xi, t) - G(s' - \xi, t)|^2 \exp(-\alpha\xi) d\xi := I_{11} - 2I_{12} + I_{22}$$

Next we deal separately with terms  $I_{11}$ ,  $I_{12}$ , and  $I_{22}$ . Denote  $a := s(t)$ ,  $b := s'(t)$ ; then by completing the square in the exponent and integrating we get

$$\begin{aligned} I_{12} &= \frac{1}{4\pi} \int t^{-1} \exp\left[-\frac{(s(t) + \xi)^2 + (s'(t) + \xi)^2}{4t} - \alpha\xi\right] d\xi \\ &= \frac{\sqrt{2\pi}}{4} t^{-1/2} \exp\left[\frac{\alpha^2 t + \alpha(b + a)}{2}\right] \exp\left[-\left(\frac{a - b}{2}\right)^2\right] \end{aligned}$$

The integrals  $I_{11}$  and  $I_{22}$  are obtained from  $I_{12}$  by replacing  $a - b$  by 0, and  $a + b$  by  $2a$  and  $2b$  respectively

$$I_{11} = \frac{\sqrt{2\pi}}{4} t^{-1/2} \exp[t\alpha^2/2 + \alpha a], \quad I_{22} = \frac{\sqrt{2\pi}}{4} t^{-1/2} \exp[t\alpha^2/2 + \alpha b]$$

Now we rewrite the expression  $I_{11} - 2I_{12} + I_{22}$  as a complete square plus a (small) correction

$$\begin{aligned} I_{11} - 2I_{12} + I_{22} &= \frac{\sqrt{2\pi}}{4} t^{-1/2} \exp\left(\frac{\alpha^2 t}{2}\right) \times \\ &\quad \left\{ \exp[\alpha a] - 2 \exp\left[\frac{\alpha(b + a)}{2}\right] \exp\left[-\left(\frac{a - b}{2}\right)^2\right] + \exp[\alpha b] \right\} \\ &= \frac{\sqrt{2\pi}}{4\sqrt{t}} \exp\left(\frac{\alpha^2 t}{2}\right) \left\{ \left[ \exp\left(\frac{\alpha a}{2}\right) - \exp\left(\frac{\alpha b}{2}\right) \right]^2 + 2 \exp\left(\frac{\alpha(b + a)}{2}\right) \left[ 1 - \exp\left[-\left(\frac{a - b}{2}\right)^2\right] \right] \right\} \end{aligned}$$

We use an elementary inequality  $|\exp(-x^2) - 1| \leq x^2$  to continue as follows:

$$\begin{aligned} &\leq \frac{\sqrt{2\pi}}{4\sqrt{t}} \exp \frac{\alpha^2 t}{2} \left\{ \underbrace{\left( \exp \frac{\alpha a}{2} - \exp \frac{\alpha b}{2} \right)^2}_{\dots} + 2 \left( \frac{a-b}{2} \right)^2 \exp \frac{\alpha(b+a)}{2} \right\} \\ &= \frac{\sqrt{2\pi}}{4\sqrt{t}} \left( \frac{a-b}{2} \right)^2 \exp \frac{\alpha^2 t}{2} \left\{ \alpha^2 \exp(\alpha\theta) + 2 \exp \frac{\alpha(b+a)}{2} \right\} \end{aligned}$$

where to estimate the difference  $\underbrace{\dots}_{\dots}$  we have used the mean value theorem with  $\theta$  between  $a$  and  $b$ . We note that

$$|a-b| = |s(t) - s'(t)| = |(\varphi - \varphi')(\theta_0)| t$$

with  $0 < \theta_0 < t$ . To finish the estimation we plug the just established estimate for  $I_1$  into (4.18) to obtain

$$\begin{aligned} |W_1| &\leq [I_{11} - 2I_{12} + I_{22}]^{1/2} \|u_0\|_\omega \\ &\leq C |(\varphi - \varphi')(\theta_0)| \sqrt{t} \exp \frac{\alpha^2 t}{4} \left\{ \alpha \exp \frac{\theta\alpha}{2} + \sqrt{2} \exp \frac{\alpha(b+a)}{4} \right\} \|u_0\|_\omega \\ &\leq C |(\varphi - \varphi')(\theta_0)| \sqrt{t} \|u_0\|_\omega \leq C \sqrt{\sigma} \|u_0\|_\omega \sup_{0 < t \leq \sigma} |\varphi - \varphi'| \end{aligned}$$

and thus

$$\sup_{0 < t \leq \sigma} |W_1| \leq A_1 \sqrt{\sigma} \|u_0\|_\omega \|\varphi - \varphi'\|_\sigma \quad (4.19)$$

As the final result we get

$$\begin{aligned} |\psi - \psi'|_\sigma &= |K\varphi - K\varphi'|_\sigma \leq L \sup_{0 < t \leq \sigma} |W_1 + W_2 + W_3| \\ &\leq L \{A_1 \sqrt{\sigma} \|u_0\|_\omega + A_2 \sigma^{3/2} + A_3 \sqrt{\sigma}\} \|\varphi - \varphi'\|_\sigma \leq C \sqrt{\sigma} \|\varphi - \varphi'\|_\sigma. \end{aligned}$$

By choosing  $\sigma$  sufficiently small we achieve the contraction. ■

### 4.3 Interface temperature

We complete the characterization of the solution on the interface by the estimate on the interface temperature. We shall also need this result later on for the differentiability of the semigroup argument.

**Proposition 5** *If the exponent  $\alpha$  in the weight satisfies  $\alpha \leq v_0$  then*

$$u(s(t), t) \leq V_0/v_0 + t^{-1/4} \|u_0\|_\omega \quad (4.20)$$

**Proof.** First, the contribution of the source term in (4.12) into the interface temperature is estimated as follows:

$$\begin{aligned} \left| \int_0^t G(s(t) - s(\tau), t - \tau) v(\tau) d\tau \right| &\leq \left| \int_0^t \frac{V_0}{2\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(s(t) - s(\tau))^2}{4(t-\tau)}\right] d\tau \right| \\ &\leq V_0 \left| \int_0^\infty \frac{2}{v_0\sqrt{\pi}} \exp(-v_0^2 \frac{\eta}{4}) d\left(\frac{v_0\sqrt{\eta}}{2}\right) \right| = V_0/v_0. \end{aligned}$$

Next we estimate the contribution from the initial data.

$$\begin{aligned} \int G(s(t) - \xi, t) u_0(\xi) d\xi &= \frac{1}{\sqrt{2t\pi}} \int \exp\left(-\frac{(s(t) - \xi)^2}{4t} + \alpha\xi\right) \exp(-\alpha\xi) u_0(\xi) d\xi \\ &\stackrel{\text{Cauchy-Schwartz}}{\leq} \|u_0\|_\omega \frac{e^{\alpha s + \alpha^2 t}}{\sqrt{2t\pi}} \left( \int \exp\left(-\frac{[s(t) - \xi + 2\alpha t]^2}{2t}\right) d\xi \right)^{1/2} \\ &= t^{-1/4} \exp(\alpha s + \alpha^2 t) \|u_0\|_\omega \end{aligned}$$

The two estimates yield the result of Proposition. ■

*From now on we shall assume that  $\alpha < v_0$ .*

## 5 Estimates for the solution in weighted spaces

In the sequel we encounter certain integrals of the error function type. To estimate them we employ the following simple result.

**Lemma 6** For  $a, b > 0$

$$\int_a^\infty \exp(-b\eta^2) d\eta \leq \begin{cases} \frac{1}{2\sqrt{b}} \exp(-ba^2), & \text{for } a > 1/\sqrt{b} \\ \frac{\sqrt{\pi}}{2\sqrt{b}}, & \text{for } 0 \leq a < 1/\sqrt{b} \end{cases}$$

**Proof.** If  $a\sqrt{b} > 1$  then

$$\begin{aligned} \int_a^\infty \exp(-b\eta^2) d\eta &= \frac{1}{\sqrt{b}} \int_{a\sqrt{b}}^\infty \exp(-\eta^2) d\eta \leq \frac{1}{\sqrt{b}} \int_{a\sqrt{b}}^\infty \eta \exp(-\eta^2) d\eta \\ &= \frac{1}{2\sqrt{b}} \exp(-ba^2) \end{aligned}$$

On the other hand, always for  $a > 0$

$$\int_a^\infty \exp(-b\eta^2) d\eta \leq \int_0^\infty \exp(-b\eta^2) d\eta = \frac{\sqrt{\pi}}{2\sqrt{b}}$$

■

Also, the following Poincaré type lemma in the weighted spaces will be instrumental below.

**Lemma 7** Let  $f \in H_\omega^1$ , then for any  $0 < c \leq 1/\alpha$

$$f^2(0) \leq \left(-\alpha + \frac{1}{c}\right) \|f\|_\omega^2 + c \|f_x\|_\omega^2 \quad (5.21)$$

**Proof.** Assume first that  $f$  is  $C^1$ . By integrating from  $-\infty$  to 0 we obtain:

$$\begin{aligned} f^2(0) &= \int_{-\infty}^0 (e^{-\alpha x} f^2)_x dx = 2 \int_{-\infty}^0 e^{-\alpha x} f_x f dx - \alpha \int_{-\infty}^0 e^{-\alpha x} f^2 dx \\ &\leq \left(-\alpha + \frac{1}{c}\right) \int e^{-\alpha x} f^2 dx + c \int e^{-\alpha x} f_x^2 dx \end{aligned}$$

By selecting  $c = 1/\alpha$  one obtains an inequality, similar to the ordinary Poincaré's inequality on a finite interval. This indicates that the role of  $1/\alpha$  is similar to the interval length for the Poincaré lemma on a finite interval. ■

## 5.1 Energy estimates

The energy estimates below are performed in the interface-attached coordinate frame (3.8); we abuse notation and still use  $x$  instead of  $\zeta$  in (3.8).

**Theorem 8** For  $0 < \alpha < v_0$ ,

$$\|u(\cdot, t)\|_{\omega}^2 \leq C + \|u_0\|_{\omega}^2 \exp(-\kappa t) \quad (5.22)$$

where

$$\kappa = \alpha(v_0 - \alpha),$$

and  $C > 0$  depends on  $\alpha$  and **sublinear** kinetics  $g$ . In addition,

$$\int_0^t \|u_x\|_{\omega}^2 dt \leq C_0 t + \|u_0\|_{\omega}^2$$

**Proof.** We multiply (3.8) by  $ue^{-\alpha x}$  and integrate by parts separately on the intervals  $(-\infty, 0)$  and  $(0, \infty)$  taking into account the interface conditions to obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_{\omega}^2 = \int e^{-\alpha x} u(u_{xx} + vu_x) dx = \frac{\alpha}{2} (v + \alpha) \|u\|_{\omega}^2 - \|u_x\|_{\omega}^2 - u(0, t)v \quad (5.23)$$

The (positive) term  $-u(0, t)v$  is due to the jump condition (3.10) at  $x = 0$ . Recall that by sublinearity (see Theorem 1)  $|v/u| = |g(u)/u| \rightarrow 0$  as  $u \rightarrow \infty$ . Therefore if  $|u| > M$ , then  $|v| < \varepsilon|u|$  for an appropriate  $M$  depending on  $\varepsilon$ . On the other hand, for  $|u| \leq M$ ,  $g(u) \leq g(M)$  (by monotonicity of  $g$ ). As a result for the last term we obtain

$$\begin{aligned} |u(0, t)| |v(t)| &= |u(0, t)| g(u(0, t)) \leq g(M)M + \varepsilon|u|^2 := K + \varepsilon|u|^2 \quad (5.24) \\ &\stackrel{(5.21)}{\leq} K + \varepsilon \left( \frac{1}{c} - \alpha \right) \|u\|_{\omega}^2 + \varepsilon c \|u_x\|_{\omega}^2 \end{aligned}$$

Substitution into (5.23) yields

$$\frac{1}{2} \frac{d}{dt} \|u\|_{\omega}^2 \leq K + \frac{1}{2} \left[ \alpha(v + \alpha) + 2\left(\frac{1}{c} - \alpha\right)\varepsilon \right] \|u\|_{\omega}^2 - (1 - \varepsilon c) \|u_x\|_{\omega}^2 \quad (5.25)$$

To optimize the estimate we set  $c = 1/\alpha$ ,  $\varepsilon = \alpha$  which results in

$$\frac{d}{dt} \|u\|_{\omega}^2 \leq 2K - \kappa \|u\|_{\omega}^2 \quad (5.26)$$

with  $\kappa = \alpha(v_0 - \alpha)$ . By Gronwall's Lemma it immediately yields

$$\|u\|_{\omega}^2 \leq \|u_0\|_{\omega}^2 \exp(-\kappa t) + \frac{2}{\kappa} K$$

To obtain the estimate on  $\|u_x\|_{\omega}^2$ , in (5.25) we set  $\varepsilon = \alpha/2$ ,  $c = 1/\alpha$ ; then (5.25) can be rearranged in the form

$$\|u_x\|_{\omega}^2 \leq 2K - \kappa \|u\|_{\omega}^2 - \frac{d}{dt} \|u\|_{\omega}^2$$

By integrating we get the integral estimate for  $\|u_x\|_{\omega}^2$  :

$$0 < \int_0^t \|u_x\|_{\omega}^2 dt \leq 2Kt - \kappa \int_0^t \|u\|_{\omega}^2 dt - \|u\|_{\omega}^2 + \|u_0\|_{\omega}^2 \leq 2Kt + \|u_0\|_{\omega}^2.$$

■

One easily arrives at the following corollary of the proof above.

**Corollary 9** *The contribution from the initial conditions decays exponentially:*

$$\|u_2(\cdot, t)\|_{\omega}^2 \leq \|u_0\|_{\omega}^2 \exp(-\kappa t) \quad (5.27)$$

**Proof.** The contribution from the initial conditions  $u_2$  is a smooth solution (no jump at zero) of the Cauchy problem for the heat equation in (3.8). For  $u_2$  the calculation similar to the one in (5.23) leads to (5.23) with the jump term  $-u(0, t)v$  missing. Thereafter, a simplified version of the proof leads to the estimate (5.27). ■

By combining the estimates on the total solution and on the contribution from the initial conditions we can easily estimate the contribution from the interface

**Corollary 10**

$$\|u_1(\cdot, t)\|_{\omega}^2 \leq \frac{4K}{\kappa} + 2 \|u_0\|_{\omega}^2 \exp(-\kappa t) \quad (5.28)$$

**Proof.**  $\|u_1(\cdot, t)\|_{\omega}^2 = \|u(\cdot, t) - u_2(\cdot, t)\|_{\omega}^2 \stackrel{\text{triangle}}{\leq} 2 \|u(\cdot, t)\|_{\omega}^2 + 2 \|u_2(\cdot, t)\|_{\omega}^2$  ■

The estimate of Theorem 8 can be made more explicit if the *kinetics function* is *bounded* instead of *sublinear*.

**Corollary 11** *For the bounded kinetics*

$$\|u(\cdot, t)\|_{\omega}^2 \leq \frac{V_0^2}{2\alpha\kappa} + \|u_0\|_{\omega}^2 \exp(-\kappa t) \quad (5.29)$$

$$\|u_1(\cdot, t)\|_{\omega}^2 \leq \frac{V_0^2}{\alpha\kappa} + 2 \|u_0\|_{\omega}^2 \exp(-\kappa t) \quad (5.30)$$

**Proof.** The estimate in (5.24) simplifies as follows:

$$|u| |v| \leq |u| V_0 \leq \frac{1}{4\alpha} V_0^2 + \alpha |u|^2$$

i.e., the constant  $K$  is replaced by  $V_0^2/(4\alpha)$ . The subsequent estimates change accordingly and lead to the results in (5.29)-(5.30). ■

**Remark 12** *If we choose, for instance  $\alpha = \lambda v_0$ , then for large  $t$  the bound becomes*

$$\|u\|_\omega^2 \leq \frac{1}{2\lambda(1-\lambda)} \frac{V_0^2}{v_0^3} + \varepsilon.$$

*This provides an estimate on the diameter of the absorbing ball (see Sec. 8 below).*

## 5.2 Spatial decay of the interface contribution

For any fixed  $t$  we show that norms of the "tails" (for  $|\zeta|$  large) of the interface contribution to the solution tend to zero, uniformly for any bounded set of interface velocities. This result is necessary for proving  $H_\omega$ -compactness of the absorbing set. For any function  $F$  we introduce the restrictions ("tails"):

$$F_M(\zeta) = \begin{cases} F(\zeta), & \zeta > M \\ 0, & \zeta < M \end{cases}, \quad F_{-M}(\zeta) = \begin{cases} F(\zeta), & \zeta < -M \\ 0, & \zeta > -M \end{cases}$$

and note that for any  $F \in H_\omega$ ,  $F_{\pm M}$  also belong to  $H_\omega$ .

**Proposition 13** *For any  $M > 1$  the norms of the restrictions of  $u_1(t, \cdot)$  to the intervals  $(-\infty, -M)$  and  $(M, \infty)$  decay exponentially in  $M$  uniformly in time:*

$$\begin{aligned} \|u_1(t, \cdot)\|_{-M} &:= \left( \int_{-\infty}^{-M} \omega(\zeta) \left[ \int_0^t G(\zeta + s(t) - s(\tau), t - \tau) v(\tau) d\tau \right]^2 d\zeta \right)^{1/2} \\ &\leq c_- \exp\left(-\frac{v_0 - \alpha}{2} M\right) \end{aligned} \quad (5.31)$$

$$\begin{aligned} \|u_1(t, \cdot)\|_M &:= \left( \int_M^\infty \omega(\zeta) \left[ \int_0^t G(\zeta + s(t) - s(\tau), t - \tau) v(\tau) d\tau \right]^2 d\zeta \right)^{1/2} \\ &\leq c_+ \exp\left(-\frac{\alpha}{2} M\right) \end{aligned} \quad (5.32)$$

where  $c_-, c_+ > 0$  depend only on the kinetics and the weight.

**Proof.** For brevity we write

$$s = s(t), \quad s' = s(\tau), \quad \Delta = s(t) - s(\tau), \quad \tilde{\tau} = t - \tau.$$

Note that by the "triangle" inequality,

$$\left\| \int_0^t G(\cdot + \Delta, t - \tau)v(\tau)d\tau \right\|_{\pm M} \leq \sup_{\tau} |v(\tau)| \int_0^t \|G(\cdot + \Delta, t - \tau)\|_{\pm M} d\tau \quad (5.33)$$

Next we evaluate the norm under the integral sign. We start with the evaluation ahead of the interface,  $\zeta < -M$ .

$$\begin{aligned} \int_{-\infty}^{-M} \omega(\zeta) G^2(\zeta + \Delta, \tilde{\tau})d\zeta &= \frac{1}{4\pi\tilde{\tau}} \int_{-\infty}^{-M} \exp\left[-\frac{(\zeta + \Delta)^2}{2(t - \tau)} - \alpha\zeta\right]d\zeta = \\ &= \frac{1}{4\pi\tilde{\tau}} \exp\left(-\frac{\Delta^2 - [\Delta + \tilde{\tau}\alpha]^2}{2\tilde{\tau}}\right) \int_{-\infty}^{-M} \exp\left[-\frac{\{\zeta + (\Delta + \tilde{\tau}\alpha)\}^2}{2\tilde{\tau}}\right]d\zeta \\ &\leq \frac{1}{2\sqrt{2\pi\tilde{\tau}}} \exp\left[\alpha\left\{\Delta + \tilde{\tau}\frac{\alpha}{2}\right\}\right] \exp\left[-\frac{\{-M + (\Delta + \tilde{\tau}\alpha)\}^2}{2\tilde{\tau}}\right] \end{aligned}$$

To obtain the last inequality we completed the square in the exponent and employed Lemma 6 (recall that  $M > 1$ , and  $\alpha - v_0 < 0$ ). Furthermore, since  $\Delta < -v_0\tilde{\tau}$  and  $(-M + \tilde{\tau}(\alpha - v_0))^2 \geq 2M\tilde{\tau}(\alpha - v_0)$  we continue as follows

$$\begin{aligned} &\leq \exp\left[\alpha\left(-v_0 + \frac{\alpha}{2}\right)\tilde{\tau}\right] \frac{1}{2\sqrt{2\pi\tilde{\tau}}} \exp\left[-\frac{\{-M + \tilde{\tau}(\alpha - v_0)\}^2}{2\tilde{\tau}}\right] \\ &\leq \frac{1}{2\sqrt{2\pi\tilde{\tau}}} \exp\left[\alpha\left(-v_0 + \frac{\alpha}{2}\right)\tilde{\tau}\right] \exp[-M(v_0 - \alpha)] \end{aligned}$$

Then integration with respect to  $\tau$  yields

$$\begin{aligned} \|u_1(t, \cdot)\|_{-M} &\stackrel{(5.33)}{\leq} \sup_{\tau} |v(\tau)| \int_0^t \left\{ \int_{-\infty}^{-M} \omega(\zeta)|G|^2 d\zeta \right\}^{1/2} d\tilde{\tau} \\ &\leq e^{-M(v_0 - \alpha)} \sup_{\tau} |v(\tau)| \int_0^t \left\{ \frac{1}{\sqrt{2\pi\tilde{\tau}}} \exp\left[-\alpha\left(v_0 - \frac{\alpha}{2}\right)\tilde{\tau}\right] \right\}^{1/2} d\tilde{\tau} \\ &\leq cV_0 e^{-M(v_0 - \alpha)/2} [\alpha(v_0 - \frac{\alpha}{2})]^{-3/4} \end{aligned}$$

For the tail *behind the interface*  $\zeta > M$  we have

$$\begin{aligned} &\frac{1}{4\pi\tilde{\tau}} \int_M^{\infty} \exp\left[-\frac{(\zeta + \Delta)^2}{2\tilde{\tau}} - \alpha\zeta\right]d\zeta \\ &\stackrel{\zeta > M}{\leq} \exp\left[-\frac{\alpha}{2}M\right] \frac{1}{4\pi} \int_{s+M}^{\infty} \tilde{\tau}^{-1} \exp\left[-\frac{(\zeta + \Delta)^2}{2\tilde{\tau}} - \frac{\alpha\zeta}{2}\right]d\zeta \\ &\leq \exp\left[-\frac{\alpha}{2}M\right] \frac{1}{4\pi} \int \tilde{\tau}^{-1} \exp\left[-\frac{(\zeta + \Delta)^2}{2\tilde{\tau}} - \frac{\alpha\zeta}{2}\right]d\zeta \\ &\leq \exp\left[-\frac{\alpha}{2}M\right] \frac{c}{\sqrt{\tilde{\tau}}} \exp\left(-\frac{\alpha}{2}\left(v_0 - \frac{\alpha}{4}\right)\tilde{\tau}\right) \end{aligned}$$

Similarly to the  $\zeta < -M$  case, integration with respect to  $\tau$  yields

$$\|u_1(t, \cdot)\|_M \leq cV_0 e^{-M\alpha/2} \left[\frac{\alpha}{2}\left(v_0 - \frac{\alpha}{4}\right)\right]^{-3/4}$$

■

## 6 Estimates for the spatial derivative of the solution

In this Section estimates on the spatial derivative of the solution are performed in the "laboratory" system of coordinates  $(x, t)$ .

**Theorem 14** . *The  $H_\omega$ -norm of the spatial derivative of the solution satisfies*

$$\|u_x(\cdot, t)\|_\omega^2 \leq \left( \frac{e^{-\alpha v_0 t}}{\sqrt{t\pi}} + \alpha\sqrt{2}e^{-\alpha(v_0-\alpha)t} \right) \|u_0\|_\omega^2 + (CV_0)^2 \left[ \frac{\alpha}{2}(v_0 - \frac{\alpha}{2}) \right]^{-3/2} \quad (6.34)$$

The principal application of the theorem is the following obvious

**Corollary 15** *For large  $t$*

$$\|u_x(\cdot, t)\|_\omega^2 \leq (CV_0)^2 \left[ \frac{\alpha}{2}(v_0 - \frac{\alpha}{2}) \right]^{-3/2} + \varepsilon := C_{ux}^2 + \varepsilon \quad (6.35)$$

uniformly in any ball  $\|u_0\|_\omega \leq R$ .

Similarly to the estimates for the solution itself, we split the proof into two parts.

### 6.1 Contribution from initial data

**Proposition 16** *The  $H_\omega$ -norm of the derivative of the contribution from initial data satisfies*

$$\|(u_2)_x(\cdot, t)\|_\omega^2 \leq \left( \frac{\exp(-\alpha v_0 t)}{\sqrt{t\pi}} + \alpha\sqrt{2} \exp\{-\alpha(v_0 - \alpha)t\} \right) \|u_0\|_\omega^2 \quad (6.36)$$

We note that the proof below is based on a curious trick that is similar to the well-known way of computing the Gaussian integral, where  $(\int_0^\infty \exp(-x^2)dx)^2$  is treated as a double integral over the quarter-plane.

**Proof.** For  $(u_2)_x(\cdot, t) = (T_2(t)u_0)_x$ , we have:

$$\begin{aligned} \|(u_2)_x(\cdot, t)\|_\omega^2 &= \int \left| \int G_x(x - \xi, t) u_0(\xi) d\xi \right|^2 \omega(x - s(t)) dx \\ &= \int \left( \int G_x(x - \xi, t) u_0(\xi) d\xi \int G_x(x - \eta, t) u_0(\eta) d\eta \right) \omega(x - s) dx \\ &= \int \left( \int_{\mathbb{R}^2} G_x(x - \xi, t) G_x(x - \eta, t) u_0(\xi) u_0(\eta) d\eta d\xi \right) \omega(x - s) dx \\ &\leq \int \left( \int_{\mathbb{R}^2} |G_x(x - \xi, t)| |G_x(x - \eta, t)| \frac{u_0(\xi)^2 + u_0(\eta)^2}{2} d\eta d\xi \right) \omega(x - s) dx \end{aligned}$$

We replace the interior double integral by the two repeated ones, choosing a convenient order of integration for each one:

$$\begin{aligned} &= \frac{1}{2} \int \omega(\xi) u_0(\xi)^2 \int \left( \int |G_x(x - \eta, t)| d\eta \right) \omega^{-1}(\xi) |G_x(x - \xi, t)| \omega(x - s) dx d\xi \\ &+ \frac{1}{2} \int \omega(\eta) u_0(\eta)^2 \int \left( \int |G_x(x - \xi, t)| d\xi \right) \omega^{-1}(\eta) |G_x(x - \eta, t)| \omega(x - s) dx d\eta \end{aligned}$$

Since

$$\int |G_x(x - \eta, t)| d\eta = \frac{1}{2\sqrt{t\pi}} \int \left| \frac{x - \eta}{2t} \right| \exp\left(-\frac{[(x - \eta)]^2}{4t}\right) d\eta = \frac{1}{\sqrt{t\pi}} \quad (6.37)$$

for the interior integrals we get

$$\begin{aligned} &\int \left( \int |G_x(x - \eta, t)| d\eta \right) \omega^{-1}(\xi) |G_x(x - \xi, t)| \omega(x - s) dx \\ &\stackrel{(6.37)}{=} \frac{1}{\sqrt{t\pi}} \int \omega^{-1}(\xi) |G_x(x - \xi, t)| \omega(x - s) dx \\ &= \frac{1}{\sqrt{t\pi}} \int \frac{|x - \xi|}{2t} \exp\left(-\frac{(x - \xi)^2}{4t} - \alpha(x - s) + \alpha\xi\right) dx \end{aligned}$$

Next we split the domain of integration into the two parts, where  $|x - \xi| = x - \xi$  and where  $|x - \xi| = -(x - \xi)$ :

$$\begin{aligned} &= \exp(\alpha s + \alpha^2 t) \int_{\xi}^{\infty} \frac{1}{\sqrt{\pi t}} \left( \frac{x - \xi + 2\alpha t}{2t} - \alpha \right) \exp\left(-\frac{[x - \xi + 2\alpha t]^2}{4t}\right) dx \\ &+ \exp(\alpha s + \alpha^2 t) \int_{-\infty}^{\xi} \frac{1}{\sqrt{\pi t}} \left( \frac{\xi - x - 2\alpha t}{2t} + \alpha \right) \exp\left(-\frac{[x - \xi + 2\alpha t]^2}{4t}\right) dx \\ &\stackrel{y=[x-\xi+2\alpha t]/(2\sqrt{t})}{=} \exp(\alpha s + \alpha^2 t) \left[ \int_{\alpha\sqrt{t}}^{\infty} \left[ \frac{1}{\sqrt{\pi t}} y \exp(-y^2) - \frac{\sqrt{2}}{\sqrt{\pi}} \alpha \exp(-y^2) \right] dy \right. \\ &\quad \left. - \exp(\alpha s + \alpha^2 t) \left[ \int_{-\infty}^{\alpha\sqrt{t}} \left[ \frac{1}{\sqrt{\pi t}} y \exp(-y^2) - \frac{\sqrt{2}}{\sqrt{\pi}} \alpha \exp(-y^2) \right] dy \right] \right) \\ &= 2 \exp(\alpha s + \alpha^2 t) \frac{1}{\sqrt{\pi t}} \left( \int_{\alpha\sqrt{t}}^{\infty} y \exp(-y^2) dy + \alpha \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\alpha\sqrt{t}} \alpha \exp(-y^2) dy \right) \\ &\leq 2 \exp(\alpha s + \alpha^2 t) \left[ \frac{\exp(-\alpha^2 t)}{2\sqrt{\pi t}} + \frac{\alpha}{\sqrt{2}} \right] \end{aligned}$$

Since  $s(t) < -v_0 t$  it yields,

$$\|(u_2)_x(\cdot, t)\|_{\omega}^2 \leq \exp(-\alpha v_0 t + \alpha^2 t) \left( \frac{\exp(-\alpha^2 t)}{\sqrt{\pi t}} + \sqrt{2}\alpha \right) \|u_0\|_{\omega}^2 \quad (6.38)$$

■

## 6.2 Contribution from the interface

**Proposition 17** *The  $H_\omega$ -norm of the derivative of the contribution from the free interface is uniformly bounded for all time*

$$\|(u_1)_x(\cdot, t)\|_\omega \leq CV_0 \left[ \frac{\alpha}{2} \left( v_0 - \frac{\alpha}{2} \right) \right]^{-3/4} \quad (6.39)$$

provided that  $\alpha < 2v_0$ .

**Proof.** From the definition of  $u_1$  and the "triangle" inequality, it is obvious that

$$\begin{aligned} \|(u_1)_x(\cdot, t)\|_\omega &\leq \int_0^t \|G_x(\cdot - s(\tau), t - \tau)\|_\omega |v(\tau)| d\tau \\ &\leq V_0 \int_0^t \|G_x(\cdot - s(\tau), t - \tau)\|_\omega d\tau \end{aligned}$$

Introduce again  $\tilde{\tau} = t - \tau$ . To estimate the norm in the integrand

$$\|G_x\|_\omega^2 = \frac{1}{4\pi} \int \frac{(x - s(\tau))^2}{4\tilde{\tau}^3} \exp \left[ -\frac{(x - s(\tau))^2}{2\tilde{\tau}} - \alpha(x - s(\tau)) \right] dx,$$

we complete the square in the exponent to  $\{(x - s(\tau) + \tilde{\tau}\alpha)\}^2/2\tilde{\tau}$ , and then directly evaluate the Gaussian integrals:

$$\begin{aligned} \|G_x\|_\omega^2 &= \left( \frac{c_1}{\tilde{\tau}^{3/2}} + \frac{c_2}{\tilde{\tau}} + \frac{c_3}{\tilde{\tau}^{1/2}} \right) \exp \left\{ \alpha \left[ s(t) - s(\tau) + \tilde{\tau} \frac{\alpha}{2} \right] \right\} \\ &\leq \left( \frac{c_1}{\tilde{\tau}^{3/2}} + \frac{c_2}{\tilde{\tau}} + \frac{c_3}{\tilde{\tau}^{1/2}} \right) \exp \left( -\alpha \left( v_0 - \frac{\alpha}{2} \right) \tilde{\tau} \right) \end{aligned}$$

From this, by using convexity of the square root, we obtain

$$\begin{aligned} \|(u_1)_x(\cdot, t)\|_\omega &\leq V_0 \int_0^t (\|G_x\|_\omega^2)^{1/2} d\tau \\ &\leq V_0 \int_0^t \left( \frac{c_1^{1/2}}{\tilde{\tau}^{3/4}} + \frac{c_2^{1/2}}{\tilde{\tau}^{1/2}} + \frac{c_3^{1/2}}{\tilde{\tau}^{1/4}} \right) \exp \left( -\frac{\alpha}{2} \left( v_0 - \frac{\alpha}{2} \right) \tilde{\tau} \right) d\tilde{\tau} \\ &\leq V_0 \left( c_1^{1/2} \Gamma(1/4) a^{-1/4} + c_2^{1/2} \Gamma(1/2) a^{-1/2} + c_3^{1/2} \Gamma(3/4) a^{-3/4} \right) \leq CV_0 a^{-3/4} \end{aligned}$$

where  $a = \frac{\alpha}{2} \left( v_0 - \frac{\alpha}{2} \right)$  and  $C$  is an absolute constant. To estimate the integrals from 0 to  $t$  above we have replaced them by the integrals from 0 to  $\infty$ . ■

This concludes the proof of Theorem 3.

## 7 Dependence of solutions on initial data

In this section we prove that solutions of the free boundary problem (3.8)-(3.10) depend continuously on the initial data; in other words, *we establish continuity of the evolution operator*. This result will be used in the sequel to demonstrate spatial smoothness of the elements of the attractor.

**Theorem 18** *Solutions of the problem (3.8)-(3.10) depend on initial conditions continuously in  $H_\omega$ . More precisely, let  $\{u(x, t), s(t)\}$ ,  $\{u'(x, t), s'(t)\}$  be solutions with initial data  $u_0, u'_0 \in B_R \subset H_\omega$ , where  $B_R = \{w : \|w\|_\omega \leq R\}$ . Then there exists  $\sigma > 0$  that depends on  $R$  alone such that for  $0 < t < \sigma$*

$$\sup_{0 < t < \sigma} |v(t) - v'(t)| < c\sigma^{-1/4} \|u_0 - u'_0\|_\omega, \quad (7.40)$$

$$\|u - u'\|_\omega < c \|u_0 - u'_0\|_\omega \quad (7.41)$$

Furthermore,

$$\int_0^t \|u_x(\cdot, \tau) - u'_x(\cdot, \tau)\|_\omega^2 d\tau \leq e^{C\sqrt{t}} \|u_0 - u'_0\|_\omega^2$$

uniformly on the ball  $B_R$ ; here  $C$  depends solely on  $\omega$ .

**Remark 19** *We state and prove continuous dependence on initial conditions only locally in time. The argument extending this result to any fixed time is based on the a priori estimates and follows closely the proof of global existence.*

**Remark 20** *If the initial data are in  $H_\omega^1$  then a similar argument yields the following estimate, analogous to (7.41):*

$$\|u(t) - u'(t)\|_{\omega,1} \leq C \|u_0 - u'_0\|_{\omega,1} \quad (7.42)$$

*This guarantees continuous dependence on the initial data in  $H_\omega^1$ .*

The proof of Theorem 18 consists of two parts. First we establish continuity of the interface velocity (7.40) and then use it to carry out the energy estimates (7.41).

## 7.1 Estimates for $v - v'$

The estimates of this section are very similar to those in Sec. 4.2; they are performed in *the laboratory coordinate system*. Introduce the norm

$$[|v - v'|]_{\sigma,1/4} = \sup_{0 < \tau < \sigma} \tau^{1/4} |v(\tau) - v'(\tau)| \quad (7.43)$$

Here the exponent  $1/4$  is selected for simplicity of presentation; it is sufficient to take  $\tau^{1/2-\varepsilon}$  for any  $0 < \varepsilon < 1/2$

Let  $v$  and  $v'$  be solutions of the integral equation in (4.11) with initial data

$u_0$  and  $u'_0$  respectively, then

$$\begin{aligned}
& |v(t) - v'(t)| = \\
& \left| g \left( \int G(s(t) - \xi, t) u_0(\xi) d\xi - \int_0^t G(s(t) - s(\tau), t - \tau) v(\tau) d\tau \right) \right. \\
& \left. - g \left( \int G(s'(t) - \xi, t) u'_0(\xi) d\xi - \int_0^t G(s'(t) - s'(\tau), t - \tau) v'(\tau) d\tau \right) \right| \\
& \leq L \left| \int G(s(t) - \xi, t) u_0(\xi) d\xi - \int G(s'(t) - \xi, t) u'_0(\xi) d\xi \right| \\
& + L \left| \int_0^t G(s(t) - s(\tau), t - \tau) v(\tau) d\tau - \int_0^t G(s'(t) - s'(\tau), t - \tau) v'(\tau) d\tau \right|,
\end{aligned}$$

where  $L$  is the Lipschitz constant of  $g$ . To continue the estimate we employ a "coordinate descent":

$$\begin{aligned}
& \leq L \left| \int (G(s(t) - \xi, t) - G(s'(t) - \xi, t)) u_0(\xi) d\xi \right| \\
& + L \left| \int G(s'(t) - \xi, t) (u_0(\xi) - u'_0(\xi)) d\xi \right| \\
& + L \left| \int_0^t G(s(t) - s(\tau), t - \tau) (v(\tau) - v'(\tau)) d\tau \right| \\
& + L \left| \int_0^t [G(s(t) - s(\tau), t - \tau) - G(s'(t) - s'(\tau), t - \tau)] v'(\tau) d\tau \right| \\
& := L(W_0 + W_1 + W_2 + W_3)
\end{aligned}$$

The term  $W_0$  is estimated as follows,

$$\begin{aligned}
|W_0(s', t)| &= \left| \int G(s', t, \xi, 0) (u_0(\xi) - u'_0(\xi)) d\xi \right| \\
&\stackrel{\text{Cauchy-Schwartz}}{\leq} \|u_0 - u'_0\|_\omega \left| \int \omega^{-1}(\xi) |G(s', t, \xi, 0)|^2 d\xi \right|^{1/2}
\end{aligned}$$

The last integral is evaluated directly

$$\int \omega^{-1} |G(s' - \xi, t)|^2 d\xi = \frac{\exp(\alpha s' + \alpha^2 t/2)}{2\pi t} \int \exp\left(-\frac{[(s' - \xi) - \alpha t]^2}{2t}\right) d\xi \leq \frac{c}{\sqrt{t}}$$

since  $\alpha s' + \alpha^2 t/2 \leq -\alpha v_0 t + \alpha^2 t/2 < 0$  for  $\alpha < 2v_0$ . Thus,

$$|W_0| \leq A_0 t^{-1/4} \|u_0 - u'_0\|_\omega$$

The term  $W_1$  is identical to  $W_1$  in (4.18) and therefore

$$|W_1| \stackrel{(4.19)}{\leq} |v(t) - v'(t)| A_1 t^{1/2} \|u_0\|_\omega$$

For the estimate of  $W_2$ , after a simple algebra we replace the exponential by 1 and integrate to obtain:

$$\begin{aligned} W_2(t) &= \left| \int_0^t \tau^{-1/4} (4\pi(t-\tau))^{-1/2} \exp \left\{ -\frac{[s(t)-s(\tau)]^2}{4(t-\tau)} \right\} \tau^{1/4} (v(\tau) - v'(\tau)) d\tau \right| \\ &\leq A_2 t^{1/4} \|v - v'\|_{\sigma, 1/4} \end{aligned}$$

Finally we estimate

$$\begin{aligned} W_3 &= \left| \int_0^t [G(s(t) - s(\tau), t - \tau) - G(s'(t) - s'(\tau), t - \tau)] v'(\tau) d\tau \right| \\ &: = \left| \int_0^t \Delta G v'(\tau) d\tau \right| \end{aligned}$$

For  $\Delta G$  we have

$$\begin{aligned} |\Delta G| &\stackrel{(4.16)}{\leq} C_0 V_0 |v(\tau') - v'(\tau')| (t - \tau)^{1/2} \\ &= C_0 V_0 |v(\tau') - v'(\tau')| (\tau')^{1/4} (t - \tau)^{1/2} (\tau')^{-1/4} \\ &\leq_{\tau \leq \tau' \leq t} C_0 V_0 |v(\tau') - v'(\tau')| (\tau')^{1/4} (t - \tau)^{1/2} \tau^{-1/4} \\ &\leq C_0 V_0 \|v - v'\|_{\sigma, 1/4} (t - \tau)^{1/2} \tau^{-1/4} \end{aligned}$$

Integration with respect to  $\tau$  yields:

$$\begin{aligned} W_3(t) &= \left| \int_0^t \Delta G v'(\tau) d\tau \right| \leq \|v - v'\|_{\sigma, 1/4} C_0 V_0^2 \int_0^t \tau^{-1/4} (t - \tau)^{1/2} d\tau \\ &= A_3 t \|v - v'\|_{\sigma, 1/4} \end{aligned}$$

We collect the estimates for  $W_0$  through  $W_3$  to obtain

$$\begin{aligned} |v(t) - v'(t)| &\leq LA_0 t^{-1/4} \|u_0 - u'_0\|_\omega + L|v(t) - v'(t)| A_1 t^{1/2} \|u_0\|_\omega \\ &\quad + LA_2 t^{1/4} \|v - v'\|_{\sigma, 1/4} + LA_3 t \|v - v'\|_{\sigma, 1/4} \end{aligned}$$

Upon multiplying this inequality by  $t^{1/4}$  and taking suprema on the interval  $(0, \sigma]$  we get:

$$\begin{aligned} \|v - v'\|_{\sigma, 1/4} &\leq LA_0 \|u_0 - u'_0\|_\omega + LA_1 \|v - v'\|_{\sigma, 1/4} \sigma^{1/2} \|u_0\|_\omega \\ &\quad + LA_2 \sigma^{1/2} \|v - v'\|_{\sigma, 1/4} + LA_3 \sigma^{5/4} \|v - v'\|_{\sigma, 1/4} \\ &\leq LA_0 \|u_0 - u'_0\|_\omega + L \left\{ A_1 \sigma^{1/2} \|u_0\|_\omega + A_2 \sigma^{1/2} + A_3 \sigma^{5/4} \right\} \|v - v'\|_{\sigma, 1/4} \end{aligned}$$

(note that all the powers of  $\sigma$  are positive). If  $\sigma$  is sufficiently small then the last inequality can be solved for  $\|v - v'\|_{\sigma, 1/4}$  yielding

$$\|v - v'\|_{\sigma, 1/4} \leq C \|u_0 - u'_0\|_\omega \quad (7.44)$$

We note that the choice of  $\sigma$  depends only on  $\|u_0\|_\omega$ ; therefore this result can be interpreted as uniformly continuous dependence of  $v$  on the initial data in any ball  $\|u_0\|_\omega \leq R$ .

## 7.2 Energy estimates for $u - u'$

In this subsection we once again abuse notation and denote by  $u(x, t)$  and  $u'(x, t)$  the two solutions of the free boundary problem in *the frame attached to the free boundary* (3.8)-(3.10) that correspond to the solutions in the laboratory coordinate system of the previous subsection. Thus  $u$  and  $u'$  solve the problems

$$u_t = u_{xx} + [u_x(0, t)]u_x, \quad g(u(0, t)) = [u_x(0, t)], \quad u(x, 0) = u_0(x),$$

$$u'_t = u'_{xx} + [u'_x(0, t)]u'_x, \quad g(u'(0, t)) = [u'_x(0, t)], \quad u'(x, 0) = u'_0(x).$$

The difference  $w = u - u'$  solves the following problem

$$\begin{aligned} w_t &= w_{xx} + [u_x(0, t)]w_x + [w_x(0, t)]u'_x; \quad w|_{t=0} = u_0 - u'_0, \\ [w_x(0, t)] &= (g(\theta))'w(0, t) = g(u(0, t)) - g(u'(0, t)), \end{aligned} \quad (7.45)$$

where  $\theta$  is an intermediate value between  $u(0, t)$  and  $u'(0, t)$ ,

We multiply the equation throughout by  $\omega w$  and integrate to obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_\omega^2 &= \int \omega w_{xx} w dx + [u_x(0, t)] \int \omega w_x w dx + [w_x(0, t)] \int \omega u'_x w dx \\ &= -[w_x]w|_0 - \|w_x\|_\omega^2 + \alpha \int \omega w_x w dx + [u_x(0, t)] \int \omega w_x w dx \\ &\quad + [w_x(0, t)] \int \omega u'_x w dx \end{aligned} \quad (7.46)$$

We need to estimate different terms in (7.46). For the first term we use Lemma 7 to obtain

$$|[w_x(0, t)]w(0, t)| \stackrel{(7.45)}{\leq} Lw(0, t)^2 \leq L \left( \frac{1}{\varepsilon} \|w\|_\omega^2 + \varepsilon \|w_x\|_\omega^2 - \alpha \|w\|_\omega^2 \right) \quad (7.47)$$

(recall that  $-(g(\theta))' \leq L$ ).

Next we note that

$$\left| \int \omega w_x w dx \right| \leq \frac{1}{2} \left( \frac{1}{\varepsilon} \|w\|_\omega^2 + \varepsilon \|w_x\|_\omega^2 \right)$$

Therefore

$$\begin{aligned} |[u_x(0, t)] \int \omega w_x w dx| &\leq \frac{1}{2} |[u_x(0, t)]| \left( \frac{1}{\varepsilon} \|w\|_\omega^2 + \varepsilon \|w_x\|_\omega^2 \right) \\ &\leq \frac{V_0}{2} \left( \frac{1}{\varepsilon} \|w\|_\omega^2 + \varepsilon \|w_x\|_\omega^2 \right) \end{aligned} \quad (7.48)$$

On the other hand, as follows from Theorem 14, for  $t \leq 1$

$$\|u'_x(\cdot, t)\|_\omega^2 \leq C_0^2 / \sqrt{t}$$

We use Cauchy-Schwartz inequality to obtain

$$\begin{aligned} \left| [w_x(0, t)] \int \omega u'_x w dx \right| &\leq L |w(0, t)| \|u'_x\|_\omega \|w\|_\omega \\ &\leq \frac{1}{2} LC_0 t^{-1/4} (\|w\|_\omega^2 + |w(0, t)|^2) \\ &\leq Ct^{-1/4} \left( \|w\|_\omega^2 + \frac{1}{\varepsilon} \|w\|_\omega^2 + \varepsilon \|w_x\|_\omega^2 - \alpha \|w\|_\omega^2 \right) \end{aligned}$$

Collecting the estimates for different terms we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_\omega^2 &\leq [-1 + \varepsilon(L + V_0/2 + \alpha/2 + Ct^{-1/4})] \|w_x\|_\omega^2 \\ &+ \left[ L \left( \frac{1}{\varepsilon} - \alpha \right) + \alpha + \frac{V_0 + \alpha}{2\varepsilon} + Ct^{-1/4} \left( 1 + \frac{1}{\varepsilon} - \alpha \right) \right] \|w\|_\omega^2 \\ &\leq -\frac{1}{2} \|w_x\|_\omega^2 + \frac{c}{2} t^{-1/2} \|w\|_\omega^2 \end{aligned}$$

where on the last step we have chosen

$$\varepsilon = \frac{1}{2(L + V_0/2 + \alpha/2) + 2Ct^{-1/4}}$$

so that the coefficient of  $\|w_x\|_\omega^2$  is  $-1/2$ ; consequently the coefficient of  $\|w\|_\omega^2$  becomes

$$c_0 + c_1 t^{-1/4} + c_2 t^{-1/2} \underset{t \leq \sigma}{\leq} \frac{c}{2} t^{-1/2}$$

We rewrite our last result as

$$\frac{d}{dt} \|w\|_\omega^2 + \|w_x\|_\omega^2 \leq ct^{-1/2} \|w\|_\omega^2. \quad (7.49)$$

From this inequality we get first that  $\frac{d}{dt} \|w\|_\omega^2 \leq ct^{-1/2} \|w\|_\omega^2$  and therefore by the Gronwall lemma

$$\|w\|_\omega^2 \leq \|w_0\|_\omega^2 \exp(2ct^{1/2});$$

which yields the first estimate in the statement of Theorem 18 . At the same time by rearranging (7.49) and integrating with respect to  $t$ , we obtain

$$\begin{aligned} \int_0^t \|w_x\|_\omega^2 d\tau &\leq \int_0^t (c\tau^{-1/2} \|w\|_\omega^2 - \frac{d}{dt} \|w\|_\omega^2) d\tau \\ &= \|w_0\|_\omega^2 \exp(2ct^{1/2}) - \|w\|_\omega^2 \leq 2 \|w_0\|_\omega^2 \exp(2ct^{1/2}) \end{aligned} \quad (7.50)$$

which is the contents of the second statement of the theorem.

**Remark 21** Obviously for  $t \geq \sigma > 0$ , the last estimate could be modified as follows

$$\int_0^t \|w_x\|_\omega^2 d\tau \leq c_1 \exp(ct) \|w_0\|_\omega^2$$

## 8 Absorbing set and attractor

In the current section, utilizing the estimates of the previous sections, we establish existence of a bounded absorbing set and of the attractor, which is compact in the weighted space  $H_\omega$ . For our purposes we rephrase Corollary 9 and the estimate in (5.30) in the following form:

**Proposition 22** (i) *The semigroup  $T_2$  is uniformly exponentially contracting in  $H_\omega$ :*

$$\|T_2(t)u_0\|_\omega \leq C \exp(-\kappa t/2) \|u_0\|_\omega,$$

for  $t \geq t_0$  and any  $t_0 > 0$  (the constant  $C$  depends on  $t_0$  for small  $t_0$ ; this dependence can be ignored for  $t_0 > 1$ ), where  $\kappa = \alpha(v_0 - \alpha) > 0$ ;

(ii) *For any  $\varepsilon > 0$ , the ball*

$$B_{abs} := \{u \in H_\omega : \|u\|_\omega \leq \frac{V_0}{\sqrt{\alpha\kappa}} + \varepsilon\}$$

*is a global absorbing set in  $H_\omega$ .*

Note that the radius of the absorbing ball reflects the contribution of the free interface alone. Next we prove that the free-interface contribution to the evolution, i.e. the family of operators  $T_1(t)$  is *uniformly compact*:

**Proposition 23** *For any  $t_0 > 0$  the orbit  $\Omega(t_0) = \cup_{t \geq t_0} T_1(t)B_r$  of any ball  $B_r = \{u_0 : \|u_0\|_\omega \leq r\}$  is relatively compact in  $H_\omega$ .*

**Proof.** We construct a finite  $\varepsilon$ -net for the orbit  $\Omega(t_0)$ . First we employ Proposition 13 and select  $M$  so that

$$\int_{|\zeta| > M} |(T_1(t)u_0)(\zeta, t)|^2 \omega(\zeta) d\zeta = \int_{|\zeta| > M} \left( \int_0^t Gv(\tau) d\tau \right)^2 \omega d\zeta \leq \varepsilon^2 \quad (8.51)$$

for all  $t > t_0$ . We have proved that  $\Omega(t_0)$  is a bounded set in  $H_\omega^1$ -norm (see (5.30) and Proposition 17), then according to Rellich's theorem on the finite interval  $[-M, M]$  the restriction of  $\Omega(t_0)$  is imbedded compactly in the corresponding restriction  $H_\omega([-M, M])$  (note that on the finite interval the weight is irrelevant, for example,  $\omega$  can be taken  $\omega \equiv 1$ ) Therefore a finite  $\varepsilon$ -net in  $H_\omega([-M, M])$  can be constructed. The estimate in (8.51) shows that for all elements of  $\Omega(t_0)$  the share of the norm that is responsible for the exterior of  $[-M, M]$  is small; therefore by extending elements of the  $\varepsilon$ -net as 0 beyond this interval we obtain an  $\varepsilon$ -net in  $H_\omega(-\infty, \infty)$ . ■

The properties of the evolution operator  $T(t)$  described in the above propositions allow us to apply the general result [26, Chap. I] that in our situation can be stated as follows:

**Theorem 24** *The  $\omega$ -limit set of the absorbing ball  $B_{abs}$ ,  $\mathcal{A} = \bigcap_{t_0 \geq 0} \overline{\Omega(t_0)}$ , is a global compact attractor for the space  $H_\omega$ ;  $\mathcal{A}$  is the maximal attractor in  $H_\omega$  and it is connected.*

Since, by definition the attractor is both positive and negative invariant with respect to the evolution, the mapping  $T(t)$  is one-to-one on the attractor. As an immediate application we obtain the following important characterization of the attractor:

**Corollary 25** *The attractor  $\mathcal{A}$  consists of  $H_\omega^1$  functions that satisfy the estimates*

$$\phi(0, t) \leq V_0/v_0 \tag{8.52}$$

$$\begin{aligned} \|\phi\|_\omega &\leq \frac{V_0}{\sqrt{2\alpha\kappa}}, \\ \|\phi_x\|_\omega &\leq B(\alpha) := CV_0\left[\frac{\alpha}{2}\left(v_0 - \frac{\alpha}{2}\right)\right]^{-3/4} \end{aligned} \tag{8.53}$$

where  $C$  is an absolute constant (a combination of some  $\Gamma$ -functions).

**Proof.** Since the mapping is onto, given  $\phi \in \mathcal{A}$  for any  $t$  there exist  $\psi \in \mathcal{A}$ , so that  $\phi = T(t)\psi$ . For  $t \rightarrow \infty$  the estimate (4.20) yields the estimate on the interface temperature (8.52). Similarly, by taking into account exponential decay of the contribution from initial data as  $t \rightarrow \infty$ , we obtain the norm estimates from (5.29) and (6.34).. ■

**Remark 26** *Since any function in the attractor can be viewed as a result of evolution by the semigroup, it therefore locally satisfies the heat equation and consequently it is locally  $C^\infty$ . In addition we can show that due to the differentiability of the velocity of the interface, functions in the attractor are  $C^3$  up to the interface.*

## 9 Evolution of the volume elements on the attractor

In this section we present the principal ingredient of the proof of finiteness of the Hausdorff dimension of the attractor. The proof is based on a study of evolution of the infinitesimal volume along the orbits in the attractor (i.e., the evolution under the linearized flow). We demonstrate that for sufficiently large  $m$  that is defined solely by the physical parameters of the problem, the  $m$ -dimensional volume decays exponentially. This property combined with the compactness suggests that the Hausdorff dimension of the attractor is no larger than  $m$ . In the arguments regarding the Hausdorff dimension we follow quite closely the ideas outlined in [26, Chap. V]. The relation between the linearized flow and the fully nonlinear evolution is considered in Sec. 10.

## 9.1 Linearized problem

We treat the problem in the interface-attached coordinate frame. Let  $U(\cdot, t)$  be an orbit in the attractor. Let us consider the formal linearization of the problem (3.8)-(3.10) about  $U$ :

$$\begin{aligned} z_t &= z_{xx} + g(U(0, t))z_x + \beta z(0, t)U_x := F'(U)z, \\ [z_x(0, t)] &= \beta z, \quad z(x, 0) = z_0(x) = U_0(x) - W_0(x), \end{aligned} \quad (9.54)$$

where

$$\beta := g'(U(0, t)) \quad (9.55)$$

Recall that  $\beta < 0$ ,  $|\beta| \leq L$  where  $L$  is the Lipschitz constant for  $g$ , and that

$$v(t) = g(U(0, t)) < 0 \quad (9.56)$$

is the interface velocity.

First we note that the linearized problem is well-posed in the following sense:

**Theorem 27** *For any  $z_0 \in H_\omega$  there exists a unique solution  $z$  of (9.54),  $z \in L^2(0, T; \Xi(t)) \cap C([0, T]; H_\omega)$  where  $\Xi(t) = \{f \in H_\omega^1, \beta f(0) + [f_x(0)] = 0\}$ , for any  $T > 0$ .*

**Proof.** This linear problem is somewhat nonstandard as it contains a non-local term  $w(0, t)$ . Nonetheless it can be handled as follows (cf. [23] where a similar problem is discussed in more detail). Consider first the problem (9.54) with a source, and zero initial conditions

$$\tilde{w}_t - [\tilde{w}_{xx} + \tilde{w}_x g(U(0, t))] = f(x, t)$$

$$\beta \tilde{w}(0, t) + [\tilde{w}_x(0, t)] = 0,$$

$$\tilde{w}(x, 0) = 0,$$

We formally rewrite the equation above as  $\mathcal{L}\tilde{w} = f(x, t)$ ; let  $\mathcal{L}^{-1}$  be its solution operator:  $\tilde{w} = \mathcal{L}^{-1}f(x, t)$ .

We regard a solution of (9.54) as a superposition  $z = \tilde{w} + W$  of an appropriate  $\tilde{w}$  and of  $W(x, t)$ , which solves the *homogeneous* problem ( $f(x, t) = 0$ ) with the initial condition  $z_0(x)$ . Then from the linearized problem we obtain an equation for  $\tilde{w}$ :

$$\tilde{w} + W = \mathcal{L}^{-1} [\beta U_x(x, t)(\tilde{w}(0, t) + W(0, t))]$$

This equation, restricted to the boundary, produces an integral equation for  $\tilde{w}(0, t)$ :

$$\mathcal{L}^{-1}[-\tilde{w}(0, t)U_x\beta + W(x, t)]_{x=0} = \tilde{w}(0, t) + W(0, t). \quad (9.57)$$

It is not difficult to show that the above equation is uniquely solvable as an integral equation with a sufficiently regular kernel. Thus,  $\tilde{w}$  is found and, consequently, the problem (9.54) can be solved locally in time.

The global existence can be obtained from the energy estimate:

$$\begin{aligned} \frac{d}{dt}E(t) &:= \frac{1}{2} \frac{d}{dt} \|z\|_\omega^2 = \int \exp(-ax) z(z_{xx} + vz_x + \beta z(0, t)U_x) dx \\ &= -\beta z^2(0, t) - \|z_x\|_\omega^2 + (\alpha - v) \int \exp(-ax) z z_x + \beta z(0, t) \int \exp(-ax) z U_x dx \end{aligned}$$

We drop the first (negative) term, use Lemma 7 and the Cauchy-Schwartz inequality to obtain

$$\begin{aligned} &\stackrel{\beta \leq L}{\leq} -\|z_x\|_\omega^2 + |\alpha - V_0| \left( \frac{1}{\varepsilon_1} \|z\|_\omega^2 + \varepsilon_1 \|z_x\|_\omega^2 \right) \\ &\quad + L \left[ \frac{1}{\varepsilon^2} \|z\|_\omega^2 + \varepsilon^2 \|z_x\|_\omega^2 - \alpha \|z\|_\omega^2 \right]^{1/2} \|z\|_\omega \|U_x\|_\omega \\ &\leq -\|z_x\|_\omega^2 + |\alpha - V_0| \left( \frac{1}{\varepsilon_1} \|z\|_\omega^2 + \varepsilon_1 \|z_x\|_\omega^2 \right) + L \left[ \frac{1}{\varepsilon} \|z\|_\omega + \varepsilon \|z_x\|_\omega \right] \|z\|_\omega C_1 \end{aligned}$$

where  $C_1$  is a bound on the  $H_\omega^1$ -norm on the elements of the attractor. By choosing  $\varepsilon$  and  $\varepsilon_1$  sufficiently small we can make the coefficient of  $\|z_x\|_\omega^2$  equal to  $-1/2$  and therefore obtain the following inequality:

$$\frac{d}{dt}E(t) + \frac{1}{2} \|z_x\|_\omega^2 \leq CE(t),$$

which yields global existence in  $L^2(0, T; \Xi(t)) \cap C([0, T]; H_\omega)$ . ■

## 9.2 Trace estimate

Now we shall estimate the evolution of the infinitesimal volume element. To this end we need to estimate the trace of the finite-dimensional projections of the generator of the linear semigroup generated by the operator  $F'$  from (9.54). Let  $\xi_j$ ,  $j = 1, \dots, m$  be elements of  $H_\omega$  and let  $z_j$  be the solution of the linearized problem with initial data  $\xi_j$ . Then the volume element spanned by  $\{\xi_1, \dots, \xi_m\}$  evolves accordingly to the well-known formula

$$|z_1(t) \wedge \dots \wedge z_m(t)| = |\xi_1 \wedge \dots \wedge \xi_m| \exp \int_0^t \text{Tr} [F'(U(\tau)) \circ Q_m(\tau)] d\tau,$$

where  $Q_m(\tau) = Q_m(\tau, U; \xi_1, \dots, \xi_m)$  is the projector onto the space  $\Xi(\tau)$  spanned by  $\{z_1(\tau), \dots, z_m(\tau)\}$ .

Let  $\phi$  be an element of  $\Xi(\tau)$ , then for the corresponding diagonal entry of the matrix of the  $m$ -dimensional projection we have

$$\begin{aligned} \langle F'\phi, \phi \rangle_\omega &= \int \omega \phi_{xx} \phi dx + v(\tau) \int \omega \phi_x \phi dx + [\phi'(0)] \int \omega U_x \phi dx \\ &= -\|\phi_x\|_\omega^2 + (\alpha + v) \int e^{-\alpha x} \phi_x \phi dx + [\phi'(0)] \int \omega U_x \phi dx \end{aligned} \quad (9.58)$$

It is easily seen via integration by parts that

$$\int \omega \phi_x \phi dx = \left(\frac{1}{2} \phi^2 e^{-\alpha x}\right)\Big|_0^\infty + \left(\frac{1}{2} \phi^2 e^{-\alpha x}\right)\Big|_{-\infty}^0 + \frac{\alpha}{2} \int \omega \phi^2 dx = \frac{\alpha}{2} \quad (9.59)$$

Thus (9.58) becomes

$$\begin{aligned} \langle F' \phi, \phi \rangle_\omega &= -[\phi'(0)]\phi(0) - \|\phi_x\|_\omega^2 + [\phi'(0)] \int \omega U_x \phi dx + \frac{\alpha^2}{2} + \frac{v\alpha}{2} \\ &\stackrel{|\beta| \leq L}{\leq} L\phi^2(0) - \|\phi_x\|_\omega^2 + \frac{\alpha^2}{2} + \frac{v\alpha}{2} + \beta\phi(0) \int \omega U_x \phi dx \end{aligned} \quad (9.60)$$

To evaluate the trace we select a special orthonormal basis in the subspace  $\Xi$ . In the choice of the basis elements we shall distinguish the two possibilities:  $\phi(0) = 0$ , which defines an  $(m-1)$ -dimensional subspace, and otherwise. Since the trace of the operator is independent of the choice of an orthonormal basis, we can choose  $m-1$  basis elements satisfying  $\phi(0) = 0$ .

We will next estimate the absolute value of the principal negative term  $-\|\phi_x\|_\omega^2$ . Since for any  $c > 0$

$$2 \int e^{-\alpha x} \phi_x \phi dx \leq c \int e^{-\alpha x} \phi_x^2 dx + \frac{1}{c} \int e^{-\alpha x} \phi^2 dx$$

then

$$\|\phi_x\|_\omega^2 \geq \frac{2}{c} \int \omega \phi_x \phi dx - \frac{1}{c^2} \stackrel{(9.59)}{=} \frac{\alpha}{c} - \frac{1}{c^2} \stackrel{\text{Set } c=2/\alpha}{=} \frac{\alpha^2}{4}$$

If  $\phi(0) = 0$  then

$$\langle F' \phi, \phi \rangle_\omega = -\|\phi_x\|_\omega^2 + \frac{\alpha^2}{2} + \frac{v\alpha}{2} \leq \frac{\alpha^2}{4} - \frac{v_0\alpha}{2} \quad (9.61)$$

we remind that the velocity is negative  $-V_0 \leq v \leq -v_0 < 0$ . Therefore for  $\phi(0) = 0$  the matrix entry is negative if  $\alpha < 2v_0$  (note that if  $\alpha = v_0$  then  $\langle F' \phi, \phi \rangle_\omega$  attains its minimum  $-v_0^2/4$ ).

If  $\phi(0) \neq 0$  we need to estimate the extra terms in (9.60) that contain  $\phi(0)$  as a factor. First we note that

$$\begin{aligned} \left| \beta\phi(0) \int \omega U_x \phi dx \right| &\leq \frac{1}{2b} \int \omega U_x^2 dx + \frac{b}{2} (\beta\phi(0))^2 \int \omega \phi^2 dx \\ &\stackrel{|\beta| \leq L}{\leq} \frac{1}{2b} \|U_x\|_\omega^2 + \frac{b}{2} (L\phi(0))^2 \end{aligned} \quad (9.62)$$

where  $b > 0$  will be chosen later. Thus, the matrix element is estimated as

follows:

$$\begin{aligned}
\mu &:= \langle F' \phi, \phi \rangle_\omega \stackrel{(9.62)}{\leq} L\phi^2(0) - \|\phi_x\|_\omega^2 + \frac{\alpha^2}{2} + \frac{v\alpha}{2} + \frac{1}{2b} \|U_x\|_\omega^2 + \frac{b}{2} L^2 \phi^2(0) \\
&\stackrel{(5.21)}{\leq} (L + L^2 \frac{b}{2}) \left( c \|\phi_x\|_\omega^2 - \alpha + \frac{1}{c} \right) - \|\phi_x\|_\omega^2 + \frac{1}{2b} \|U_x\|_\omega^2 + \frac{\alpha(\alpha - v_0)}{2} \\
&\stackrel{\alpha < v_0}{\leq} [(L + L^2 \frac{b}{2})c - 1] \|\phi_x\|_\omega^2 + (L + L^2 \frac{b}{2}) \frac{1}{c} + \frac{1}{2b} \|U_x\|_\omega^2
\end{aligned}$$

To eliminate the term with  $\|\phi_x\|_\omega^2$ , which we cannot control, we select  $b = 2/L$ ,  $c = 1/(2L)$  and obtain

$$\mu \leq 4L^2 + \frac{L}{4} \|U_x\|_\omega^2 \stackrel{(8.53)}{\leq} 4L^2 + \frac{L}{4} B(\alpha)$$

Thus the *diagonal entry for the basis element with  $\phi(0) \neq 0$  is bounded above.*

If for definiteness  $\alpha$  is taken as  $\alpha = \lambda v_0$ ,  $0 < \lambda < 1$ , then

$$\mu \leq C(\lambda) \frac{V_0^2 L}{4v_0^3} + 4L^2 \tag{9.63}$$

where  $C(\lambda) = O(1)$ . In the leading order, assuming  $v_0 \ll 1$ ,

$$\mu \leq CLV_0^2/v_0^3, \tag{9.64}$$

where  $C$  is an absolute constant. Employing the above estimates for the trace entries (9.61), (9.63) we complete the estimate for the trace

$$\text{Tr} [F'(U(\tau), V(\tau)) \circ Q_m(\tau)] = \sum_{i=1}^m \langle F' \phi_i, \phi_i \rangle_\omega \leq \mu - (m-1)v_0^2/4$$

In the leading order,

$$m \stackrel{(9.64)}{>} CLV_0^2/v_0^5 \tag{9.65}$$

is sufficient for the trace to become negative.

Thus, the linearized evolution contracts the infinitesimal  $m$ -dimensional volume. This indicates that the Hausdorff dimension should be less than  $CLV_0^2/v_0^5$ , provided the *nonlinear* evolution also contracts the elementary  $m$ -dimensional volume. Therefore, to justify an upper bound for Hausdorff dimension, we need to demonstrate the differentiability of the solution operator with respect to the initial conditions.

## 10 Differentiability of the semigroup

To utilize the trace estimate developed in the previous section we need to demonstrate that the nonlinear evolution of the volume is well approximated by the

evolution generated by the linearization (9.54). This will be ensured by the theorem on differentiability of the evolution operator with respect to the initial conditions (cf. [26, Sec. V.3.3]):

**Theorem 28** *Let  $U$  and  $W$  be two orbits  $U = T(t)U_0$ ,  $W = T(t)W_0$ ,  $U_0, W_0 \in H_{\omega,1}$  and  $z(t)$  be a solution of the linearized problem (9.54) with the initial condition  $z(0) = U_0 - W_0$ . Then for any  $t_0$ ,*

$$\|U(t) - W(t) - z(t)\|_{\omega} \leq \text{const} \|U_0 - W_0\|_{\omega}^{3/2}$$

where  $0 \leq t \leq t_0$  and the constant depends only on  $t_0$ .

Note that the mapping  $z(0) \rightarrow z(t)$  is the Fréchet differential of  $T(t)$  at the point  $U_0$ .

**Proof.** First we note that by Proposition 5 the interface temperature is uniformly bounded, and for solutions in the attractor (4.20) yields

$$U(0, t) \leq V_0/v_0. \quad (10.66)$$

Then since  $g(U)$  is strictly monotone,  $g(U) \leq V_M := g(V_0/v_0) < V_0$ . The function  $(g^{-1})'(v)$  is bounded on the interval  $[v_0, V_M]$ , consequently  $g'(U(0, t)) \geq d_1$ . Also we assume  $g \in C^2$  and therefore  $|g''|$  is bounded on  $[0, V_0/v_0]$  by some constant  $d_2$ .

It is easy to see that the difference  $w = U - W$  solves the following problem

$$\begin{aligned} w_t &= w_{xx} + g(U(0, t))w_x - [w_x(0, t)]w_x + [w_x(0, t)]U_x, \\ [w_x(0, t)] &= -\Delta v = g(U(0, t)) - g(W(0, t)) = g'(\theta)w(0, t) \\ w(x, 0) &= U_0(x) - W_0(x). \end{aligned} \quad (10.67)$$

which is similar to (7.45). Let  $z(x, t)$  be a solution of the linearization in (9.54). Then the difference  $y = w - z$  is a solution of the following problem

$$\begin{aligned} y_t &= y_{xx} + g(U(0, t))y_x + [y_x(0, t)]U_x - [w_x(0, t)]w_x, \\ [y_x]_{x=0} &= g'(\theta)w|_{x=0} - g'(U(0, t))z|_{x=0} = [g'(\theta) - g'(U)]w|_{x=0} + g'(U)y|_{x=0} \end{aligned} \quad (10.68)$$

$$y(x, 0) = 0$$

where  $\theta$  is between  $U(0, t)$  and  $W(0, t)$ . We denote

$$G := \{g'(\theta) - g'(U)\} w|_{x=0}$$

then the boundary condition in (10.68) takes the form

$$y(0, t) = \{[y_x]_{x=0} - G\} / g'(U(0, t)) \quad (10.69)$$

Now we carry out an energy estimate. We multiply the equation in (10.68) by  $\omega y$  and integrate the first term by parts to obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y\|_{\omega}^2 &= -[y_x]y|_0 - \|y_x\|_{\omega}^2 + \alpha \int \omega y_x y dx \\ + [U_x(0, t)] \int \omega y_x y dx &+ [y_x(0, t)] \int \omega U_x y dx - [w_x(0, t)] \int \omega w_x y dx \end{aligned} \quad (10.70)$$

For the first term above we have

$$\begin{aligned} |[y_x(0, t)] y(0, t)| &\stackrel{(10.69)}{=} |Gy(0, t) + g'y^2(0, t)| \leq \{G^2 + y^2(0, t)\}/2 + |g'| y^2(0, t) \\ &\stackrel{\text{Lemma 7}}{\leq} (|g'| + 1/2) + G^2/2 \end{aligned}$$

Next,

$$\alpha \int \omega y_x y dx \leq \frac{\alpha}{2}$$

Similarly,

$$\left| [U_x(0, t)] \int \omega y_x y dx \right|_{\|U_x\| \leq V_0} \leq V_0 \frac{1}{2} \left( \frac{1}{\varepsilon} \|y\|_\omega^2 + \varepsilon \|y_x\|_\omega^2 \right)$$

For the next to the last term in (10.70) we obtain

$$\begin{aligned} &\left| [y_x(0, t)] \int \omega U_x y dx \right| \\ &\stackrel{\text{Cauchy-Schwartz}}{\leq} (|g'| \|y(0, t)\| + |G|) \int \omega |U_x y| dx \leq (|g'| \|y(0, t)\| + |G|) \|y\|_\omega \|U_x\|_\omega \\ &\stackrel{\text{Lemma 7}}{\leq} \left\{ |g'| \left( \frac{1}{\varepsilon} \|y\|_\omega^2 + \varepsilon \|y_x\|_\omega^2 \right)^{1/2} + |G| \right\} \|y\|_\omega \|U_x\|_\omega \end{aligned}$$

We use convexity of the square root function and the inequality  $2ab \leq a^2 + b^2$  to continue

$$\begin{aligned} &\leq \left\{ |g'| (\sqrt{1/\varepsilon} \|y\|_\omega + \sqrt{\varepsilon} \|y_x\|_\omega) + |G| \right\} \|y\|_\omega \|U_x\|_\omega \\ &\leq |g'| \sqrt{1/\varepsilon} \|y\|_\omega^2 \|U_x\|_\omega + \frac{|g'|^2}{2} \sqrt{\varepsilon} \|y_x\|_\omega^2 + \frac{1}{2} \|y\|_\omega^2 \|U_x\|_\omega^2 + \frac{1}{2} \|y\|_\omega^2 + \frac{1}{2} G^2 \|U_x\|_\omega^2 \\ &:= \frac{|g'|^2}{2} \sqrt{\varepsilon} \|y_x\|_\omega^2 + A \|y\|_\omega^2 + \frac{B^2}{2} G^2 \end{aligned}$$

where  $B$  is an upper bound on  $\|U_x\|_\omega$  [cf. (8.53)], and  $A$  is a positive constant, dependent on  $\varepsilon$ .

Finally for the last term in (10.70)

$$\begin{aligned} \left| \int \omega w_x [w_x(0, t)] y dx \right| &\leq |[w_x(0, t)]| \|w_x\|_\omega \|y\|_\omega \leq \frac{c}{2} \|w_x\|_\omega^2 + \frac{1}{2c} |[w_x(0, t)]|^2 \|y\|_\omega^2 \\ &= \frac{c}{2} \|w_x\|_\omega^2 + \frac{1}{2c} \|y\|_\omega^2 |\Delta v|^2 \stackrel{(7.44)}{\leq} \frac{c}{2} \|w_x\|_\omega^2 + \frac{1}{c} \|y\|_\omega^2 C_1 \|w_0\|_\omega^2 t^{-1/2} \end{aligned}$$

Upon selecting  $c = \|w_0\|_\omega^2$  we estimate this term as

$$\left| \int \omega w_x [w_x(0, t)] y dx \right| \leq \frac{1}{2} \|w_x\|_\omega^2 \|w_0\|_\omega^2 + C_1 t^{-1/2} \|y\|_\omega^2$$

Collecting the estimates for all the terms in (10.70) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y\|_\omega^2 &\leq (g' + 1/2) \left( \frac{1}{\varepsilon} \|y\|_\omega^2 + \varepsilon \|y_x\|_\omega^2 \right) + G^2/2 - \|y_x\|_\omega^2 \\ &\quad + \frac{\alpha}{2} \left( \frac{1}{\varepsilon} \|y\|_\omega^2 + \varepsilon \|y_x\|_\omega^2 \right) + V_0 \left( \frac{1}{\varepsilon} \|y\|_\omega^2 + \varepsilon \|y_x\|_\omega^2 \right) \\ &\quad + \frac{|g'|^2}{2} \sqrt{\varepsilon} \|y_x\|_\omega^2 + A \|y\|_\omega^2 + \frac{B^2}{2} G^2 + \frac{1}{2} \|w_x\|_\omega^2 \|w_0\|_\omega^2 + C_1 t^{-1/2} \|y\|_\omega^2 \end{aligned} \quad (10.71)$$

Now we can select  $\varepsilon$  sufficiently small so that the combined coefficient by  $\|y_x\|_\omega^2$

$$-1 + \frac{|g'|^2}{2} \sqrt{\varepsilon} + \varepsilon(g' + \frac{1}{2} + \frac{1}{2}\alpha + V_0)$$

is nonpositive.

We collect the like terms in (10.71), drop the nonpositive term with  $\|y_x\|_\omega^2$  and rearrange to obtain

$$\frac{1}{2} \frac{d}{dt} \|y\|_\omega^2 - (C + C_1 t^{-1/2}) \|y\|_\omega^2 \leq \frac{B^2}{2} G^2 + \frac{1}{2} \|w_x\|_\omega^2 \|w_0\|_\omega^2.$$

Recall that  $|g''| \leq d_2$ , hence

$$|G|^2 = |[g'(\theta) - g'(U(0, t))]w|^2 \leq d_2^2 |\theta - U(0, t)|^2 |w(0, t)|^2 \leq d_2^2 |w(0, t)|^4$$

Note that  $|w(0, t)|$  is uniformly bounded. Indeed as follows from (8.52), on the attractor  $\sup |w(0, t)| \leq 2V_0/v_0 = C_0$ . Therefore  $|w(0, t)|^4 \leq C_0 |w(0, t)|^3$ . Here instead of the exponent 3 we could use  $4 - \varepsilon$ ,  $0 < \varepsilon < 2$ , with the corresponding change in the constant  $C_0$  (in order to demonstrate the differentiability we need to produce an exponent *larger* than 2; on the other hand to have an integrable singularity at  $t = 0$ , the exponent needs to be less than 4). We continue the estimate as follows

$$|G|^2 \leq C_1 |w(0, t)|^3 \stackrel{(10.67)}{\leq} \frac{C_1}{d_1^3} |\Delta v|^3 \stackrel{(7.44)}{\leq} C \|w_0\|_\omega^3 t^{-3/4}$$

As a result we arrive at the following differential inequality

$$\frac{1}{2} \frac{d}{dt} \|y\|_\omega^2 - (C + C_1 t^{-1/2}) \|y\|_\omega^2 \leq E \|w_0\|_\omega^3 t^{-3/4} + \frac{1}{2} \|w_x\|_\omega^2 \|w_0\|_\omega^2. \quad (10.72)$$

By the Gronwall lemma,

$$\begin{aligned} \|y\|_\omega^2 &\leq \exp(Ct + \frac{2}{3} C_1 t^{1/2}) \left( E \|w_0\|_\omega^3 \int_0^t t^{-3/4} d\tau + \frac{1}{2} \|w_0\|_\omega^2 \int_0^t \|w_x\|_\omega^2 d\tau \right) \\ &\stackrel{(7.50)}{\leq} \exp(Ct + \frac{2}{3} C_1 t^{1/2}) (4E \|w_0\|_\omega^3 t^{1/4} + \frac{1}{2} \|w_0\|_\omega^4 \exp(Ct)) \stackrel{\|w_0\|_\omega < 1}{\leq} c \|w_0\|_\omega^3 \end{aligned}$$

where  $c$  is a bound for the time dependent factor for  $t \leq T$ . This completes the proof of Theorem 28. ■

Finally, the estimate for the dimension of the linear volume element and differentiability of the semigroup yield the principal result of the paper:

**Theorem 29** *The Hausdorff dimension of the attractor  $\mathcal{A}$  is finite,*

$$\dim_H \mathcal{A} \leq CLV_0^2/v_0^5$$

In conclusion we note that the estimate exhibits a transparent dependence of the dimension on the properties of the kinetics.

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