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Please answer all 4 questions, showing your work in reasonable detail. One sheet of notes may be used for the test. Books, papers and **calculators** are not permitted.

1. (9 pts) (a) Solve the heat equation  $u_t = 2u_{xx}$  on the interval  $0 \leq x \leq \pi$  with boundary conditions  $u(0, t) = u(\pi, t) = 0$  and the initial conditions  $u(x, 0) = 2 \sin 3x$ . Determine how long it takes for the magnitude of the solution to fall below 50% of the initial magnitude (please leave the answer in terms of logarithms).

1(b)(4pts.) Indicate whether the given function is even, odd, or neither:

(i)  $x^3 - 2x$ , \_\_\_\_\_ (ii)  $x^2 - 2x \sin x + 1$ , \_\_\_\_\_ (iii)  $|x|^3 \sin^2 x$ , \_\_\_\_\_ (iv)  $e^x$  \_\_\_\_\_.

2. (a) (8pts.) Find all the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, y'(5) = 0.$$

I.e., solve the problem and find for which  $\lambda$  it has nonzero solutions  $y$ . Assume  $\lambda > 0$ .

(b) (5 pts) The function  $u(x, y)$  satisfies the equation  $u_{xx} + u_{yy} + u_x = 0$ . If  $u(x, y) = X(x)Y(y)$ , find two ordinary differential equations for  $X$  and  $Y$ .

3. For the function

$$f(x) = \begin{cases} 0, & 0 < x < 2, \\ 1, & 2 < x < 4. \end{cases}$$

consider a Fourier series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

(a)(5 p.) Sketch the function to which the series converges on the interval  $-5 < x < 10$

(b)(7 p.) Find the coefficients  $b_n$  of the series above.

4. Consider the system

$$\frac{dx}{dt} = (2+x)(x-y), \quad \frac{dy}{dt} = (x-4)(y+x)$$

(a) (2p) Verify that  $(-2, 2)$  is a critical point of the system.

(b) (5p) Find the corresponding linear system near the critical point.

(c) (3p) Find the eigenvalues of the linear system. What can you conclude about the behavior of the nonlinear system near the critical point?

(d) (2p) Find other critical point(s) of the system.