

_____ Please
answer all 7 questions, showing your work in detail. Comment **One sheet of notes is permitted for the test.**

1.(12p.) (a) Solve the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2} + 1, \quad y(\pi) = 0.$$

(b) Is the equation linear or nonlinear?

(c) On what interval is the solution valid

2.(a)(6p.) The plot below shows graphs of solutions of an equation of the form $dy/dt = f(y)$. (a) What is the nature of solutions $y = 1$ and $y = 3$? (b) Give an example of $f(y)$ so that solutions of $dy/dt = f(y)$ have the graphs such as those shown.

2.(b)(9p.) Find an implicit algebraic equation for the solution of the problem,

$$\frac{dy}{dx} = \frac{e^x}{y+1}, \quad y(0) = 0.$$

3. (a)(7p.) Find the general (real) solution of the equation

$$u'' - 2u' + 2u = 0.$$

(b)(8p.) Find a particular solution of

$$u'' - 2u' + 2u = 2 + 2t$$

4. Suppose $f(x)$ has a Fourier series,

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right)$$

where

$$f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 3. \end{cases}$$

(a)(8p.) Find the coefficients c_n of the series above.

(b)(5p.) Denote the sum of the series in part (a) by f . Use properties of the series (periodic, even or odd) to sketch its graph and determine to which values the series converges at the following points

$$f(1/2) = \quad , f(-1/2) = \quad , f(1) = \quad , f(4) = \quad , f(-2) = \quad , f(-6) =$$

5.(a)(6p.) Find the eigenvalues and eigenvectors of the matrix $C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

(b)(7p.) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as eigenvectors with eigenvalues -3 and -1

correspondingly. Find the solution of the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(c)(7p.) Sketch a phase portrait of the system $\mathbf{x}' = A\mathbf{x}$ of part (b).

6. Consider the nonlinear system

$$x' = (-1 + y)x - \frac{1}{2}x, \quad y' = (1 - x)y - \frac{1}{2}y.$$

(a)(5p.) Find all the equilibrium solutions (critical points of the system).

(b)(5p.) Find the linear system that approximates the given system in the neighborhood of $(0, 0)$, and determine the type and stability of $(0, 0)$.

7. Consider the heat conduction problem

$$u_t = u_{xx}, \quad 0 < x < 10, \quad t > 0,$$

$$u_x(0, t) = 10, \quad t > 0$$

$$u(10, t) = 50, \quad t > 0$$

$$u(x, 0) = 40 - 4x, \quad 0 < x < 10$$

(a)(7p.) Find the steady state temperature distribution $v(x)$ that will be approached as $t \rightarrow \infty$.

(b)(3p.) Let $u(x, t) = v(x) + w(x, t)$, where v is the steady solution from part (a), i.e., let w be the transient temperature distribution. **Write down** the problem satisfied by w (i.e., the equation, the initial and boundary conditions). **Do not solve** the problem for w you obtain in this question.

(d)(5p.) Use the method of separation of variables to replace $u_{xx} + xu_t = 0$ by a pair of ordinary differential equations. Comment