2001 solutions.

1. $y' + \frac{2}{t} y = \frac{\cos t}{t^2} + 1 \qquad y(\pi) = 0$

a) Linear!

Integrating factor t^2

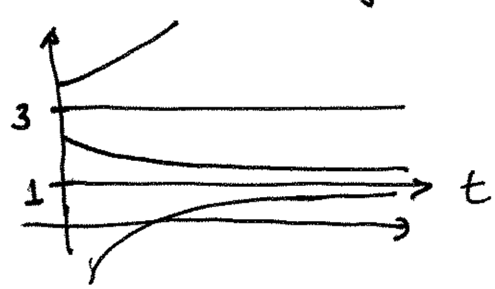
$(t^2 y)' = \cos t + t^2$

$t^2 y - t^2 y\big|_{t=\pi} = \int_{\pi}^{t} (\cos t + t^2)\, dt = \sin t + \frac{t^3}{3}$

$- \frac{\pi^3}{3} \Rightarrow y = \frac{1}{t^2}\left[\sin t + \frac{t^3 - \pi^3}{3} \right]$

b) Solution is valid for $0 < t < \infty$

2. a) The missing plot:



1 and 3 are stable and unstable equilibria

$\frac{dy}{dt} = k\,(y-1)(y-3)\ , \quad k > 0$

b) $\frac{dy}{dx} = \frac{e^x}{y+1} \qquad y(0) = 0$

$\int_0^y (y+1)\, dy = \int_0^x e^x\, dx \Rightarrow \frac{y^2}{2} + y = e^x - 1$

3. a) General real solution of $u'' - 2u' + 2u = 0$

$u = c_1 e^t \cos t + c_2 e^t \sin t$

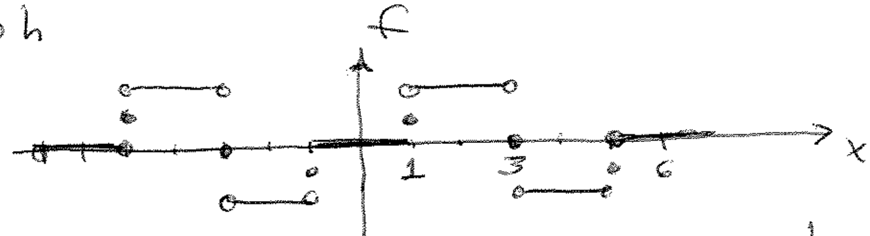
b) $u'' - 2u' + 2u = 2 + 2t$; $U = 2 + t$ (undeterm. coef's)

4. $f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{3}$, $f = \begin{cases} 0, & 0<x<1 \\ 1, & 1<x<3 \end{cases}$

$$c_n = \frac{2}{3} \int_1^3 \sin \frac{n\pi x}{3}\, dx = \frac{-2}{n\pi} \cos \frac{n\pi x}{3} \Big|_1^3$$

$$= -\frac{2}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{3}\right] = -\frac{2}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{3}\right]$$

graph



$f(\frac{1}{2}) = 0$, $f(-\frac{1}{2}) = 0$, $f(1) = \frac{1}{2}$, $f(4) = -1$

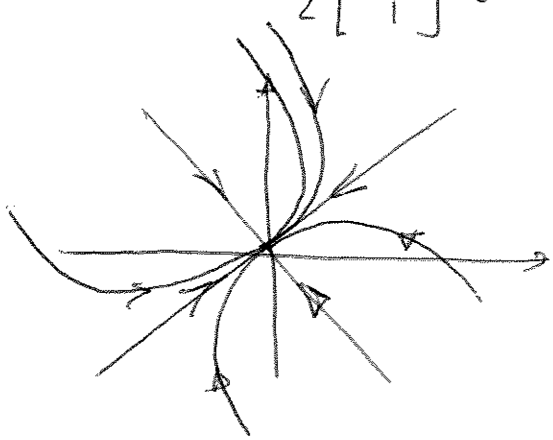
$f(-2) = -1$, $f(-6) = 0$.

5. a) $C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $\lambda_1 = -2$, $\lambda_2 = 0$

$\xi^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $x = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

$x(t) = \frac{5}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \Rightarrow x(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$



Phase portrait

6. $x' = (-1+y)x - \frac{1}{2}x$; $y' = (1-x)y - \frac{1}{2}y$

$\left[(-1+y) - \frac{1}{2}\right]x = 0 \qquad y\left(1-x-\frac{1}{2}\right) = 0$

$y = \frac{3}{2}, x = 0 \longrightarrow$

$\boxed{\begin{array}{l} y=0, x=0 \\ x = \frac{1}{2}, y = \frac{3}{2} \end{array}}$

$y\left(\frac{1}{2}-x\right) = 0$

$x = \frac{1}{2} \longrightarrow y = \frac{3}{2}$

$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{bmatrix} -3/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow$ The critical point is a saddle (unstable!)

7. a) $u_t = u_{xx}, 0 < x < 10, t > 0$; $u_x(0,t) = 10$, $u(10,t) = 50$,

$u(x,0) = 40 - 4x, \quad 0 < x < 10$

Steady solution $v'' = 0 \quad v'(0) = 10, v(10) = 50$

$\rightarrow v(x) = 10x - 50$.

$u(x,t) = w(x,t) + v(x)$. Then

$w_t = w_{xx}, \quad w_x(0,t) = 0, \quad w(10,t) = 0, \quad w(x,0) = -14x + 9($

$(= u(x,0) - v(x))$

b) $u_{xx} + xu_t = 0, \quad u(x,t) = X(x)T(t)$

$$\frac{X''(x)}{xX(x)} = -\frac{T'(t)}{T(t)} = \lambda \Rightarrow X'' - \lambda x X = 0,$$

$$T' + \lambda T = 0.$$