MATH-2400 Exam #3 Answers
Intro. Diff. Equations December 7, 2004

Please answer all 4 questions, showing your work in reasonable detail. One sheet of notes may be used for the test. Books, papers and **calculators** are not permitted.

1. (9 pts) (a) Solve the heat equation $u_t = 2u_{xx}$ on the interval $0 \leq x \leq \pi$ with boundary conditions $u(0,t) = u(\pi,t) = 0$ and the initial conditions $u(x,0) = 2\sin 3x$. Determine how long it takes for the magnitude of the solution to fall below 50% of the initial magnitude (please leave the answer in terms of logarithms).

$$\underline{u(x,t) = 3\exp(-2\cdot 3^2 t)\sin 3x}$$

$\exp(-18t) = 1/2 \Rightarrow \underline{t = \frac{1}{18}\ln 2}$

1(b)(4pts.) Indicate whether the given function is even, odd, or neither:

(i) $x^3 - 2x$, odd (ii) $x^2 - 2x\sin x + 1$, even (iii) $|x|^3 \sin^2 x$, even (iv) e^x neither.

2. (a) (8pts.) Find all the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, y'(5) = 0.$$

I.e., solve the problem and find for which λ it has nonzero solutions y. Assume $\lambda > 0$.

$\underline{\lambda_n = (\frac{n\pi}{5})^2; y_n(x) = \cos\frac{n\pi x}{5}, n = 1, 2, 3...}$ Note that $n = 0$ also gives a solution $\lambda_0 = 0; y_0(x) = 1$, it is excluded by considering $\lambda > 0$.

(b) (5 pts) The function $u(x,y)$ satisfies the equation $u_{xx} + u_{yy} + u_x = 0$. If $u(x,y) = X(x)Y(y)$, find two ordinary differential equations for X and Y.

$$\underline{X'' + X' - cX = 0; Y'' + cY = 0}$$

3. For the function

$$f(x) = \begin{cases} 0, & 0 < x < 2, \\ 1, & 2 < x < 4. \end{cases}$$

consider a Fourier series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{4})$$

(a)(5 p.) Sketch the function to which the series converges on the interval $-5 < x < 10$.

The graph should be both **odd and periodic, period 8.**

(b)(7 p.) Find the coefficients b_n of the series above.

$$b_n = \frac{2}{4}\int_0^4 f(x)\sin(\frac{n\pi x}{4})dx = \frac{1}{2}\int_2^4 \sin(\frac{n\pi x}{4})dx = \underline{\frac{2}{n\pi}(\cos(\frac{n\pi}{2}) - (-1)^n)}$$

4. Consider the system

$$\frac{dx}{dt} = (2+x)(x-y) := F, \frac{dy}{dt} = (x-4)(y+x) := G$$

(a) (2p) Verify that $(-2,2)$ is a critical point of the system.

$$\boxed{(2+x)(x-y)|_{(-2,2)} = 0; (x-4)(y+x)|_{(-2,2)} = 0}$$

(b) (5p) Find the corresponding linear system near the critical point.

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}_{(-2,2)} = \begin{bmatrix} -4 & 0 \\ -6 & -6 \end{bmatrix}$$

The linear system:

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

(c) (3p) Find the eigenvalues of the linear system. What can you conclude about the behavior of the nonlinear system near the critical point? Eigenvalues: $-4, -6$; stable node(d)

(2p) Find other critical point(s) of the system. Other critical points $(0,0), (4,4)$