Final Exam Math Analysis, Fall 2005

Part I is worth 60 points, and part II is worth 40 points. You can work on both parts with your books closed, but you must hand in part I before you open your book

Part I. Notes, books and calculators are not allowed.

1.(a) Define a countable set.

(b) Use *the definition* to prove that the set $A = \{(m, n)|m, n \text{ integers}\}$ is countable.

(c) Can a sequence of real numbers be uncountable?

2.(a) Define a Cauchy sequence.

(b) Use *the definition* to prove that the sequence $x_n = \frac{1}{n} \sin n, n = 1, 2, ...$ is a Cauchy sequence.

3.(a) Define a limit point of a set. Give an example of a closed set with exactly one limit point.

3.(b) Let f be defined on a set S. Define $\lim_{x \to a} f(x)$.

3.(c) Define open set. Use the definition to prove that the set $\{x \in \mathbb{R}|x \neq 0\}$ is open.

4.Give examples of the following:

- A nested sequence of closed sets $E_1 \subset E_2 \subset E_3 \subset ...$ such that $\cap E_n = \emptyset$.
- A continuous function that is not differentiable at a point.
- An integrable function, for which the fundamental theorem of calculus doesn't hold.
- A bounded continuous function that does not attain its minimum.
- A family of open sets, whose intersection is nonempty and closed.

5.(a) Define what it means for a function to be differentiable at x_0.

(b) Use *the definition* to prove that the function

$$g(x) = \begin{cases} |x|^{3/2} \sin \frac{1}{x}, \text{ for } x \neq 0 \\ 0, \text{ for } x = 0 \end{cases}$$

is differentiable at $x = 0$ and that $g'(0) = 0$.

(b) Prove that g' is not continuous.

6. For each of the following statements indicate whether they are true or false. For those which are false give counterexamples.

1. Every Cauchy sequence is bounded.
2. If $g(x)$ is differentiable for all $x \in \mathbb{R}$ then for any c there exist a and b so that
$$g'(c) = \frac{g(b) - g(a)}{b - a}$$
3. If any point x of the set E is its limit point then E is closed.
4. The function $f(x) = \sqrt{x}$ is uniformly continuous on the set $D = [1, \infty)$.

5. If one point p is removed from an open set E, then $E - \{p\}$ is open.

6. If one point p is removed from a closed set B, then $B - \{p\}$ is never closed.

Part II

Open Book. Notes and calculators are not allowed. Before you start using the textbook, you must hand in part I of the exam. Select **4** out of the following 5 problems.

7. Let f be Riemann integrable. Prove that the function F defined by $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$.

8. Let $x_0 = a > 1$, and define $x_n = \sqrt{x_{n-1}}$, for $n = 1, 2, 3, ...$ Prove that the sequence $\{x_n\}$ converges.

9. Let $f(x) \geq 0$ be continuous on [0,1] and $\int_0^1 f(x)dx = 0$. Prove that $f(x) \equiv 0$.

10. Let f be continuous function on (a, b). Suppose f is (i) monotone increasing, and (ii) bounded on (a, b). Prove that f is uniformly continuous on (a, b). Show that if any of the assumptions (i-ii) is dropped, then f is not necessarily uniformly continuous.

11. Use the *definition of uniform continuity* to prove that $f(x) = x^2$ is uniformly continuous on $[0, 11]$.