Mathematical Analysis I. Math-4200, Fall 2006
Extra credit problems

E1. Prove that any set of non-intersecting letters $\top$ of arbitrary sizes and orientations in the plane is countable. Is a similar statement true for letters C?

E2. It is easy to prove (by using the Cantor–Bernstein theorem) that the sets $[0,1)$ and $(0,1)$ have the same cardinality. Construct a one-to-one correspondence between these two sets explicitly.

E3. Demoted.

E4. For every sequence k_0, k_1, $k_2, \ldots$ of nonnegative integers let

$$x_n = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\cfrac{\cdots}{\cdots + \cfrac{1}{k_{n-1} + \cfrac{1}{k_n}}}}}}$$

be the sequence of continued fractions. Prove that x_0, x_1, $x_2, \ldots$ is a Cauchy sequence and that any positive real number can be approximated by such a *continued fraction* (if the integers are selected in an appropriate fashion). [Strichartz: Sec. 2.3.3, problem 5.]

E5. Give an example of two closed sets of real numbers A and B such that $A+B$ is not closed. Here $A+B=\{c : c=a+b,\ a\in A,\ b\in B\}$.

E6. Every rational x can be written in the form $x=a/b$, where $b>0$, and a and b are integers without any common divisors. When $x=0$, we take $b=1$. Consider the function χ (chi) defined on $\mathbb{R}$ by

$$\chi(x) = \begin{cases} 1/b, & \text{for } x = a/b \\ 0, & \text{for } x \text{ irrational} \end{cases}$$

Determine points of continuity and discontinuity of $\chi(x)$.

E7. Let f be defined on a finite open interval (α, β) and uniformly continuous. Show that the limit of f exists at the end points and f can be extended to a uniformly continuous function on the closed interval $[\alpha, \beta]$.

E8. Give an example of a function on $\mathbb{R}$ that has the intermediate value property for every interval (it takes on all values between $f(a)$ and $f(b)$ on $a \le x \le b$) but fails to be continuous at a point. Can such a function have jump discontinuities? [Strichartz: Sec. 4.5.4, problem 17.]

E9. Show that if $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, there exists an absolutely convergent series $\sum_{k=1}^{\infty} b_k$ such that $\lim_{k\to\infty} a_k/b_k = 0$. Explain why this result shows that there is no "universal" comparison series for testing absolute convergence.