Math-6200 Real Analysis Spring 2004.
Fourier Series: an outline

The goal is to outline a proof of the following fact:
The exponentials $\{e_n(x) := \exp(inx),\ n \in Z\}$ *form an orthonormal basis in* $L^2([-\pi,\pi))$ with the measure $\mu = \lambda/(2\pi)$
The proof contains the two basic ingredient which are important per se:

- Continuous functions are dense in L^2. A more general result will be established, based on properties of convolutions.
- Any continuous function on $[0,1]$ can be approximated by trigonometric polynomials, i.e. by finite linear combinations of $e_n(x)$. This follows easily from the Bernstein theorem on approximation by polynomials that was proved in class via Law of large numbers

Convolutions on R^n

Let f and g be measurable with respect to Lebesgue measure dx. Define

$$(f * g)(x) = \int f(y)g(x-y)dx$$

Young's Theorem *Assume* $p, q, r \in [1, \infty]$ *with*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

If $f \in L^p$, $g \in L^q$ *then* $f * g$ *is defined a.e. and* $f * g \in L^r$. *moreover,*

$$||f * g||_r \leq ||f||_p ||g||_q$$

In particular in L^1, $||f * g||_1 \leq ||f||_1 ||g||_1$
If $q = p'$, (where $1/p + 1/p' = 1$) then $r = \infty$ and even stronger result holds:
Theorem *If* $f \in L^p$, $g \in L^{p'}$ *then* $f * g$ *is defined for all* x *and is bounded and uniformly continuous on* R^n.

Approximate identity

Theorem *Let* $\phi_k \in L^1$ *and*
1. $\lim_{k\to\infty} \int \phi_k \, dx = c$.
2. $\int |\phi_k| \, dx \leq M$
3. For all $r > 0$,

$$\lim_{k\to\infty} \int_{|x|\geq r} \phi_k dx = 0$$

Then for any $f \in L^p(R^n)$, $1 \leq p < \infty$,

$$\lim_{k\to\infty} ||f * \phi_k - cf||_p = 0$$

If in addition ϕ_k *are chosen smooth,* $\phi_k \in C^N$, *then* $f * \phi_k \in C^N$ *as well.*
This theorem allows us approximate any $f \in L^p$ by smooth or continuous functions. Convolution with a smooth approximate identity is called a *mollification.*