MATH-4100-01 October 1, 2007 Exam #1 **Name:**______________________________

Please answer all **4** questions, showing your work in reasonable detail. **Closed** books; laptops and **calculators** are not permitted.

1 (30 pts.) (a) For the system $Ax = b$ reduce $[A\ b]$ to upper triangular form U

$$[A \quad b] = \begin{bmatrix} 3 & 5 & 1 & b_1 \\ 3 & 3 & 1 & b_2 \\ -3 & -7 & -1 & b_3 \end{bmatrix} \to [U \quad c]$$

(b) Factor the 3×3 matrix A into LU.

(c) Find a basis in the column space $C(A)$. What condition(s) on the vector b allow the system $Ax = b$ to be solvable?

(d) Write down the complete solution of $Ax = [1,\ 1, -1]^T$ (the 3rd column of A).

Solution (a)-(b)

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} : [A \quad b] \to \begin{bmatrix} 3 & 5 & 1 & b_1 \\ 0 & -2 & 0 & b_2 - b_1 \\ 0 & -2 & 0 & b_3 + b_1 \end{bmatrix};$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} : [L_1 A \quad L_1 b] \to \begin{bmatrix} 3 & 5 & 1 & b_1 \\ 0 & -2 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{bmatrix}$$

Factorization

$$A = (L_2 L_1)^{-1} \begin{bmatrix} 3 & 5 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) A has two pivots, therefore $C(A)$ is spanned by its 2 columns which are not proportional, say by $[5, 3, -7]^T$ and $[1, 1, -1]^T$; $C(A)$ is a plane in $\mathbb{R}^3$. A description of this plane through an equation follows from the solvabitlity condition: $b \in C(A)$ if $b_3 - b_2 + 2b_1 = 0$. This is the equation for $C(A) : 2x_1 - x_2 + x_3 = 0$.

(d) The null space of A is the same is the null space of the echelon matrix U. It is obtained by setting the free variable $x_3 = 1$ and finding the pivot variables from the equations: $\mathbf{x}_{null} = x_3 \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$. Since $A\mathbf{x} = \mathbf{col}_3$ a particular solution can be taken $\mathbf{x}_p = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_{null}$

2 (25 pts.) **(a)** Let A be any matrix and x, y, two vectors. **True or False**: If Ax and Ay are linearly independent then x and y are linearly independent.

Solution (a) True, if they were linearly dependent, $\lambda x + \mu y = 0$, then $\lambda A x + \mu A y = 0$

(b) Suppose that row operations (elimination) reduce the matrices A and B to the same row echelon form

$$R = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Which of the four subspaces are sure to be the same for A and B? $[C(A) \stackrel{?}{=} C(B),\ C(A^T) \stackrel{?}{=} C(B^T),\ N(A) \stackrel{?}{=} N(B),\ N(A^T) \stackrel{?}{=} N(B^T)]$

(ii) Each time the subspaces in part (i) are the same for A and B, find a basis for that subspace.

Solution (b) The row spaces are the same $C(A^T) = C(B^T) = C(R^T)$, also $N(A) = N(B) = N(R)$. A basis for $C(R^T)$: $[1, 1, 0, 4], [0, 2, 1, 2]$. A basis for the nullspace: $[1/2, -1/2, 1, 0], [-3, -1, 0, 1]$.

3 (25 pts.) Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Find the nullspace of A. **(b)** What is the rank of A?

(c) Give the complete solution to $Ax = \begin{bmatrix} 2 \\ 6 \\ -6 \end{bmatrix}$

Solution (a) $N(A) = N(U)$, where $A = LU$. It is easier to deal with the RREF of U

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

that also has the same null-space as A. There are two free variables x_3 and x_5; $N(R)$ is spanned by $x_3\,[-1, -2, 1, 0, 0]^T + x_5\,[-1, 1, 0, -1, 1]^T$.

(b) $\text{rank}(A) = 3$ (the number of pivots).

(c) Find a particular solution by setting the free variables to $0 : \mathbf{x}_p = [x_1, x_2, 0, x_4, 0]^T$

$$\begin{aligned} A\mathbf{x}_p &= LU\mathbf{x}_p = L \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \mathbf{x}_p \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & -7 & 1 \end{bmatrix} \begin{bmatrix} x_1 + 4x_4 \\ x_2 + 2x_4 \\ x_4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \end{aligned}$$

Do forward substitution to obtain

$$x_1 + 4x_4 = 2, \quad x_2 + 2x_4 = 0, \quad x_4 = 0$$

and $\mathbf{x}_p = [2, 0, 0, 0, 0]^T$. This answer is also clear from the fact that the right-hand side of the equation is twice the first column of L.

The complete solution is

$$\mathbf{x} = [2, 0, 0, 0, 0]^T + x_3\,[-1, -2, 1, 0, 0]^T + x_5\,[-1, 1, 0, -1, 1]^T$$

4 (20 pts.) **(a)** Find a matrix A such that $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ x+2y-3z \end{bmatrix}$.

(b) Suppose $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $B = \mathbf{u}\mathbf{v}^T$. Find special solutions of $B\mathbf{x} = \mathbf{0}$. Give geometric descriptions of $C(B)$ and $N(B)$.

Solution. **(a)**

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -3 \end{bmatrix}$$

(b) $B\mathbf{x} = \mathbf{u}\mathbf{v}^{\mathbf{T}}\mathbf{x} = \mathbf{0}$, $\text{rank}(B) = 1$. It is clear that vectors for which $\mathbf{v}^{\mathbf{T}}\mathbf{x} = \mathbf{0}$,

$$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

are solutions of $B\mathbf{x} = \mathbf{0}$. In the equation above x_2 and x_3 are free variables. Any solution of $B\mathbf{x} = \mathbf{0}$ is then given by $x_2\,[-2,1,0]^T + x_3\,[-2,0,1]^T$.

Since all the columns of B are proportional to $\mathbf{u}$, $C(B) = \text{span}(\mathbf{u})$. Geometrically $C(B)$ is the line through the origin with the direction vector $\mathbf{u}$, while $N(B)$ is the plane (again through the origin) perpendicular to $\mathbf{v}$.