Polylinear Algebra and Volumes

In these notes through a series of exercises I explain, why for the $n \times n$ matrix formed by n vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}, ..., \mathbf{z}$ in $\mathbb{R}^n$ the determinant is equal to the volume of the parallelepiped spanned by the vectors $w, \mathbf{x}, \mathbf{y}, ..., \mathbf{z}$.

Let $\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^n$ be a basis in the space $V = \mathbb{R}^n$.

1. A real-valued function of m vector arguments $l(\mathbf{x}, \mathbf{y}, ..., \mathbf{z})$, $\mathbf{x}, \mathbf{y}, ..., \mathbf{z} \in V$ is called an m-linear form if it is linear in any of its arguments, i.e.,

$$l(c_1\mathbf{x}_1 + c_2\mathbf{x}_2, \mathbf{y}, ..., \mathbf{z}) = c_1 l(\mathbf{x}_1, \mathbf{y}, ..., \mathbf{z}) + c_2 l(\mathbf{x}_2, \mathbf{y}, ..., \mathbf{z})$$

   with the similar equations for all the m independent vector variables.

   (a) The set of all m-linear forms with operations of addition and multiplication by scalars is a vector space itself. It is denoted by $V' \otimes V' \otimes ... \otimes V'$. This space is called covariant m-valent tensors.

   (b) Find its basis and dimension. [Start with 1-linear and bilinear forms. Examples $l(\mathbf{x}) = x_3$ is a one-linear form; $s(\mathbf{x}, \mathbf{y}) = x_3 y_3$is bilinear]

2. An m-linear form l is called antisymmetric if it changes sign if any two if its arguments switch places

$$l(..., \mathbf{a}, ..., \mathbf{b}, ...) = -l(..., \mathbf{b}, ..., \mathbf{a}, ...)$$

   Examples of antisymmetric forms:

   (a) i. $\phi(\mathbf{x}, \mathbf{y}) = x_1 y_3 - x_3 y_1$
       ii. In $\mathbb{R}^3$, $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \det(\mathbf{xyz})$

   (b) Prove that the set A_m of m-linear antisymmetric forms is a vector space

   (c) Define $l_{ij} := \mathbf{e}^i \wedge \mathbf{e}^j$ as follows: $l_{ij}(\mathbf{e}^i, \mathbf{e}^j) = 1 = -l_{ij}(\mathbf{e}^j, \mathbf{e}^i)$ and $l_{ij}(\mathbf{e}^p, \mathbf{e}^q) = 0$, if p and q are not a permutation of i, j. On other vectors the value of l_{ij} is obtained from the values on the basis vectors by linearity

       i. Determine $(\mathbf{e}^i \wedge \mathbf{e}^j)(\mathbf{x}, \mathbf{y})$. Show that $\mathbf{e}^i \wedge \mathbf{e}^j$, $i < j$, form a basis in the space A_2 of bilinear antisymmetric forms.
       ii. Find a basis and dimension for A_m, $2 < m \leq n$

3. In particular, the space of antisymmetric n-forms (where $n = \dim V$) is one-dimensional. Therefore any antisymmetric n-form ϕ is proportional to the basis form $\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge ... \wedge \mathbf{e}^n$:

$$\phi(\mathbf{w}, \mathbf{x}, \mathbf{y}, ..., \mathbf{z}) = c\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge ... \wedge \mathbf{e}^n(\mathbf{w}, \mathbf{x}, \mathbf{y}, ..., \mathbf{z})$$

   where c is a constant.

(a) Show that the *signed* volume of the parallelepiped spanned by n vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}, ..., \mathbf{z}$ is an antisymmetric n-form. Therefore

$$Vol(\mathbf{w}, \mathbf{x}, \mathbf{y}, ..., \mathbf{z}) = c\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge ... \wedge \mathbf{e}^n(\mathbf{w}, \mathbf{x}, \mathbf{y}, ..., \mathbf{z}) = c \det(\mathbf{w}, \mathbf{x}, \mathbf{y}, ..., \mathbf{z}).$$

(b) By comparing Vol and det on some special n-tuple of vectors show that $c = 1$.