MATH-4100-02 October 29, 2007 Exam #2 Solutions

Please answer all **6** questions, showing your work in reasonable detail. **Closed** books; laptops and **calculators** are not permitted.

1
2
3
4
5
6

1 (19 pts.) Suppose the $n \times n$ matrix A_n has 5s along its main diagonal and 2s along the diagonal below and in its $(1, n)$ position:

$$A_4 = \begin{bmatrix} 5 & 0 & 0 & 2 \\ 2 & 5 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

Find by cofactors of row 1 or otherwise the determinant of A_4 and then $\det A_n$ for $n > 4$.

By cofactors $5 * 5^3 - 2 * 2^3$,similarly for $n \times n : 5 * 5^{n-1} + (-1)^{n-1} 2 * 2^{n-1} = 5^n + (-1)^{n-1} 2^n$

2 (10 pts.) If you know that $\det A = 7$, what is $\det B$?

$$A = \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{bmatrix}, \quad B = \begin{bmatrix} \text{row 3} + \text{row 2} + \text{row 1} \\ \text{row 2} + \text{row 1} \\ \text{row 1} \end{bmatrix}$$

$$\begin{aligned} \det \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{bmatrix} &= -\det \begin{bmatrix} \text{row 3} \\ \text{row 2} \\ \text{row 1} \end{bmatrix} = -\det \begin{bmatrix} \text{row 3} \\ \text{row 2} + \text{row 1} \\ \text{row 1} \end{bmatrix} \\ &= -\det \begin{bmatrix} \text{row 3} + \text{row 2} + \text{row 1} \\ \text{row 2} + \text{row 1} \\ \text{row 1} \end{bmatrix} \end{aligned}$$

Therefore $\det B = -\det A = -7$

3 (19 pts.) We are looking for the parabola $y = \alpha + \beta t + \gamma t^2$ that gives the least squares fit to the five measurements given in the table:

t	-2	-1	0	1	2
y	1	1	1	0	2

(a) Write down the system of five equations (not solvable!) for the parabola $\alpha + \beta t + \gamma t^2$ to go through those five points.

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

(b) *Write down* equations you would solve to find the best α, β, γ. DO **NOT** SOLVE FOR α, β, γ.

$A^T A x = A^T y$:

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 13 \end{bmatrix}$$

4 (19 pts.) Find the matrix of projection onto the plane $x + y - z = 0$. Find the projection of $\mathbf{b}$=$[2, 1, 2]^T$ onto the plane. Hint: It is helpful to select a basis in the plane.

To select a basis solve

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one pivot; solutions

$$A = \begin{matrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned}
A^T A &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \left(A^T A\right)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \\
\mathbf{p} &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 5/3 \\ 2/3 \\ 7/3 \end{bmatrix}
\end{aligned}$$

The easier way is to project $\mathbf{b}$ onto the normal to the plane $\mathbf{n} = [1, 1, -1]^T$. This projection is

$$\mathbf{n}\mathbf{n}^T\mathbf{b}/\mathbf{n}^T\mathbf{n} = \frac{1}{3}\mathbf{n}$$

Then

$$\mathbf{p} = \mathbf{b} - \frac{1}{3}\mathbf{n} = [5/3, 2/3, 7/3]^T$$

5 (19 pts.) Given vectors

$$\mathbf{q}_1 = \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 1/3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) Apply Gram-Schmidt to get orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$

(b) After a little thinking find the projection of $\mathbf{a}_3$ onto the orthogonal complement of $\mathbf{q}_1$ and $\mathbf{a}_2$.

$\mathbf{q}_2 = \mathbf{a}_2/\sqrt{2}$, $\tilde{q}_3 = \mathbf{a}_3 - (\mathbf{a}_3 \cdot \mathbf{q}_1)\,\mathbf{q}_1 - (\mathbf{a}_3 \cdot \mathbf{q}_2)\,\mathbf{q}_2 = \mathbf{a}_3 - (\mathbf{4/3})\,\mathbf{q}_1 + \left(\mathbf{1}/\sqrt{2}\right)\mathbf{q}_2 =$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 8/9 \\ 0 \\ 8/9 \\ 4/9 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7/18 \\ 1 \\ -7/18 \\ 14/9 \end{bmatrix}$$

$\|\tilde{q}_3\| = \frac{1}{9}(2 * \frac{7^2}{4} + 14^2 + 1)^{1/2} = \frac{1}{18}\sqrt{443}\sqrt{2}$

(b) this projection is $\tilde{q}_3$ itself.

6 (14 pts.)(a) The matrix below has orthogonal columns

$$W = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

Without much computation find W^{-1}.

$$WW^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

therefore

$$W^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

(b) Solve the least squares problem $V\mathbf{x} = [1, 2, 3, 4]^T$, where V is the 4×2 matrix, formed by the first two columns of W in part (a). $V^T V x = V^T b$,

$$2Ix = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, x = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}$$