PHYS6520 *Quantum Mechanics II*

Spring 2002 Problem Set #1

Due at Start of Class on January 28

1. (Merzbacher Exercise 16.1.) In the spin matrix formalism, show that if and only if the expectation value of a physical quantity A is real-valued, the matrix A is Hermitian. Prove, but direct calculation, that the eigenvalues of any Hermitian 2×2 matrix are real and its eigenspinors orthogonal if the two eigenvalues are different. What happens if they are the same?

2. (Merzbacher Exercise 16.3.) Starting with an infinitesimal rotation about the unit vector $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$, prove that the rotation matrix R can be represented as

$$R = \exp(-\phi\hat{n} \cdot \vec{X})$$

where

$$X_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad X_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad X_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are three antisymmetric matrices. Work out their commutation relations and compare them with the commutation relations for the components of angular momentum.

3. (See Merzbacher Exercises 16.6, 16.7, and 16.8.)

   a. Prove that the Pauli matrices are unitary, and that

   $$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$$

   b. Prove that

   $$\sigma_x\sigma_y = i\sigma_z \qquad \sigma_y\sigma_z = i\sigma_x \qquad \sigma_z\sigma_x = i\sigma_y$$

   and that any two different Pauli matrices anticommute, that is $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$, and so forth.

   c. Prove that the only matrix which commutes with all three Pauli matrices is a multiple of the identity.

4. (Merzbacher Exercise 16.9.) Take advantage of properties (16.54) and (16.55) [These are equations in Merzbacher for the products of Pauli matrices (see above) and that the traces of Pauli matrices are all zero.] of the Pauli matrices to work out the eigenvalues and eigenspinors of a matrix A expressed as

$$A = \lambda_0 1 + \vec{\lambda} \cdot \vec{\sigma}$$

Specialize to the case $\lambda_0 = 0$ and $\lambda = \hat{n}$, where $\hat{n}$ is a real-valued arbitrary unit vector.

5. (Merzbacher Exercise 16.13.) Show that no operator of the form $\vec{L} + a\vec{\sigma}$, other than $\vec{J} = \vec{L} + (\hbar/2)\vec{\sigma}$, commutes with the scalar $\vec{\sigma} \cdot \vec{L}$.

6. (Merzbacher Exercise 16.15.) In many applications, conservation laws and selection rules cause a decaying two-level system to be prepared in an eigenstate of σ_z, say $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and governed by the simple normal Hamiltonian matrix

$$H = a1 + b\sigma_x$$

where a and b are generally complex constants. In terms of the energy difference $\Delta E = E_{02} - E_{01}$ and the decay rates Γ_1 and Γ_2, calculate the probabilities of finding the system at time t in state α or state β, respectively.