PHYS6510 *Quantum Mechanics I*

Fall 2001 Problem Set #2
Due at Start of Class on September 24

1. Rutherford's experiment showed that the size of the atomic nucleus is on the order of a few fm. (One fm$= 10^{-15}$m.) Atomic and optical physics experiments show that the energy of electrons bound in an atom is on the order of several eV. Use the uncertainty principle to show that these observations are inconsistent with the notion that electrons are bound inside the nuclei of atoms.

2. (Sakurai 1.4) Using the rules of bra-ket algebra, prove or evaluate the following:

   a. $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, where X and Y are operators;

   b. $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators;

   c. $\exp[if(A)] = ?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known;

   d. $\sum_{a'} \psi^\star_{a'}(\vec{x}')\psi_{a'}(\vec{x}'')$, where $\psi_{a'}(\vec{x}') = \langle \vec{x}' | a' \rangle$.

3. (Sakurai 1.5)

   a. Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle$, $\langle a''|\alpha\rangle$, ... and $\langle a'|\beta\rangle$, $\langle a''|\beta\rangle$, ... are all known, where $|a'\rangle$, $|a''\rangle$, ... form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

   b. We now consider a spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|s_z = \hbar/2\rangle$ and $|s_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (S_z diagonal) basis.

4. (Sakurai 1.10) The Hamiltonian operator for a two-state system is given by

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

   where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

5. (Sakurai 1.14) A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   a. Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?

   b. Give a physical example where all this is relevant.

6. (Sakurai 1.18c; See Problem Set #1.) Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet is given by

$$\langle x'|\alpha\rangle = (2\pi^2 d)^{-1/4} \exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x' - \langle x\rangle)^2}{4d^2}\right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}$$

Prove that the requirement

$$\langle x'|\Delta x|\alpha\rangle = (\text{imaginary number})\langle x'|\Delta p|\alpha\rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with your result from Problem Set #1.

7. (Sakurai 1.26) Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle\langle a^{(r)}|$$

8. (Sakurai 2.3) An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\vec{S}\cdot\hat{\vec{n}}$ with eigenvalue $\hbar/2$, where $\hat{\vec{n}}$ is a unit vector, lying in the xz-plane, that makes an angle β with the z-axis.

   a. Obtain the probability for finding the electron in the $s_x = \hbar/2$ state as a function of time.
   b. Find the expectation value of S_x as a function of time.
   c. For your own peace of mind, show that your answers make good sense in the extreme cases (i) $\beta \to 0$ and (ii) $\beta \to \pi/2$.