On the Global Solution of Linear Programs with Linear Complementarity Constraints

J. E. Mitchell[1] J. Hu[1] J.-S. Pang[2]
K. P. Bennett[1] G. Kunapuli[1]

[1] Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

[2] Industrial and Enterprise Systems Engineering Department
UIUC, Urbana, IL 61801

INFORMS Seattle
November 4-7, 2007

Abstract

This talk presents a parameter-free integer-programming based algorithm for the global resolution of a linear program with linear complementarity constraints (LPCC). The cornerstone of the algorithm is a minimax integer program formulation that characterizes and provides certificates for the three outcomes—infeasibility, unboundedness, or solvability—of an LPCC. Computational results demonstrate that the algorithm can handle infeasible, unbounded, and solvable LPCCs effectively.

Outline

1. Introduction and Motivation
2. MIP formulation of LPCCs
3. Global solution framework
   - Master Problem
   - Cuts for the Master Problem
4. Refinements
   - Cut Sparsification
5. Computational Results
   - Feasible LPCCs
   - Unbounded LPCCs
   - Infeasible LPCCs
   - Box-constrained quadratic programs
6. Conclusions

Standard form LPCC

Let $c \in \mathcal{R}^n$, $d \in \mathcal{R}^m$, $f \in \mathcal{R}^k$, $q \in \mathcal{R}^m$, $A \in \mathcal{R}^{k \times n}$, $B \in \mathcal{R}^{k \times m}$, $M \in \mathcal{R}^{m \times m}$, and $N \in \mathcal{R}^{m \times n}$ be given.
Find $(x, y) \in \mathcal{R}^n \times \mathcal{R}^m$ to globally solve the linear program with complementarity constraints (LPCC):

$$\begin{array}{ll} \underset{(x,y)}{\text{minimize}} & c^T x + d^T y \\ \text{subject to} & Ax + By \;\geq\; f \\ \text{and} & 0 \;\leq\; y \;\perp\; q + Nx + My \;\geq\; 0, \end{array}$$

(We write “LPCC” but say “LPEC” because it is easier to pronounce.)

Preliminary observations

An LPCC is equivalent to 2^m linear programs, each called a piece and derived from a subset $\alpha \subseteq \{1, \cdots, m\}$ with complement $\bar{\alpha}$:

$$\text{LP}(\alpha): \quad \boxed{\begin{array}{ll} \underset{(x,y)}{\text{minimize}} & c^T x + d^T y \\[1em] \text{subject to} & Ax + By \geq f \\[1em] & (\, q + Nx + My \,)_{\alpha} \geq 0 = y_{\alpha} \\[1em] \text{and} & (\, q + Nx + My \,)_{\bar{\alpha}} = 0 \leq y_{\bar{\alpha}} \end{array}}$$

Thus, there are 3 states of an LPCC in general:

- infeasibility–all pieces are infeasible
- unboundedness–one piece is feasible and unbounded below
- global solvability–one piece is feasible and all feasible pieces are bounded below.

Preliminary observations

An LPCC is equivalent to 2^m linear programs, each called a piece and derived from a subset $\alpha \subseteq \{1, \cdots, m\}$ with complement $\bar{\alpha}$:

$$\text{LP}(\alpha): \quad \boxed{\begin{array}{ll} \underset{(x,y)}{\text{minimize}} & c^T x + d^T y \\[2ex] \text{subject to} & Ax + By \geq f \\[2ex] & (\, q + Nx + My \,)_{\alpha} \geq 0 = y_{\alpha} \\[2ex] \text{and} & (\, q + Nx + My \,)_{\bar{\alpha}} = 0 \leq y_{\bar{\alpha}} \end{array}}$$

Thus, there are 3 states of an LPCC in general:

- infeasibility–all pieces are infeasible
- unboundedness–one piece is feasible and unbounded below
- global solvability–one piece is feasible and all feasible pieces are bounded below.

Preliminary observations

An LPCC is equivalent to 2^m linear programs, each called a piece and derived from a subset $\alpha \subseteq \{1, \cdots, m\}$ with complement $\bar{\alpha}$:

$$\mathrm{LP}(\alpha): \quad \boxed{\begin{array}{ll} \underset{(x,y)}{\text{minimize}} & c^T x + d^T y \\ \text{subject to} & Ax + By \geq f \\ & (\, q + Nx + My \,)_\alpha \geq 0 = y_\alpha \\ \text{and} & (\, q + Nx + My \,)_{\bar{\alpha}} = 0 \leq y_{\bar{\alpha}} \end{array}}$$

Thus, there are 3 states of an LPCC in general:

- infeasibility–all pieces are infeasible
- unboundedness–one piece is feasible and unbounded below
- global solvability–one piece is feasible and all feasible pieces are bounded below.

Goals

To develop a finite-time algorithm to resolve an LPCC in one of its 3 states, without complete enumeration of all the pieces and without any a priori assumptions and/or bounds.

To provide certificates for the respective states at termination:

- *an infeasible piece, if LPCC is infeasible*
- *an unbounded piece, if LPCC is feasible but unbounded below*
- *a globally optimal solution, if it exists.*

To leverage the state-of-the-art advances in linear and integer programming.

Fundamental importance

The LPCC plays the same important role in disjunctive nonlinear programs as a linear program does in convex programs.

Additionally, it has many applications of its own:

Novel paradigms in mathematical programming

- hierarchical optimization
- inverse optimization

Key formulations for

- B-stationary conditions of MPECs
  - verification and computation without MPEC-constraint qualification
- global resolution of nonconvex quadratic programs

Outline

1. Introduction and Motivation
2. MIP formulation of LPCCs
3. Global solution framework
    - Master Problem
    - Cuts for the Master Problem
4. Refinements
    - Cut Sparsification
5. Computational Results
    - Feasible LPCCs
    - Unbounded LPCCs
    - Infeasible LPCCs
    - Box-constrained quadratic programs
6. Conclusions

Equivalent Integer Program

Given a sufficiently large parameter θ and denoting the vector of ones by **1**, get an equivalent mixed integer problem:

$$\begin{array}{lrcl}
\underset{(x,y,z)}{\text{minimize}} & c^T x + d^T y & & \\
\text{subject to} & Ax + By & \geq & f \\
 & \theta\, z \geq q + Nx + My & \geq & 0 \\
 & \theta(\mathbf{1} - z) \geq y & \geq & 0 \\
\text{and} & z & \in & \{0,1\}^m
\end{array}$$

Dual problem for a fixed z

Dual $\mathrm{DP}(\theta; z)$:

$$\begin{array}{ll} \underset{(\lambda, u^{\pm}, v)}{\text{maximize}} & f^T\lambda + q^T(u^+ - u^-) - \theta \left[z^T u^+ + (\mathbf{1} - z)^T v \right] \\ \text{subject to} & A^T\lambda - N^T(u^+ - u^-) = c \\ & B^T\lambda - M^T(u^+ - u^-) - v \leq d \\ \text{and} & (\lambda, u^{\pm}, v) \geq 0, \end{array}$$

whose feasible region, assumed nonempty throughout,

$$\Xi \equiv \left\{ (\lambda, u^{\pm}, v) \geq 0 : \begin{array}{l} A^T\lambda - N^T(u^+ - u^-) = c \\ B^T\lambda - M^T(u^+ - u^-) - v \leq d \end{array} \right\}$$

is independent of θ.
Note: $\Xi \neq \emptyset \Leftrightarrow \exists\,(\lambda, u)$ with $\lambda \geq 0$ such that $A^T\lambda + N^T u = c$.

Removing the parameter θ

• Any feasible solution (x^0, y^0) of the LPCC induces a pair (θ_0, z^0), where $\theta_0 > 0$ and $z^0 \in \{0,1\}^m$, such that the pair (x^0, y^0) is feasible to the LP(θ, z^0) for all $\theta \geq \theta_0$, and

$$\begin{aligned} (\, q + Nx^0 + My^0 \,)_i \; > \; 0 \quad &\Rightarrow \quad z_i^0 \; = \; 1 \\ (\, y^0 \,)_i \; > \; 0 \quad &\Rightarrow \quad z_i^0 \; = \; 0. \end{aligned}$$

• Conversely, if (x^0, y^0) is feasible to the LP(θ, z^0) for some $\theta \geq 0$, then (x^0, y^0) is feasible to the LPCC.

• If (x^0, y^0) is an optimal solution to the LPCC, then it is optimal to the LP(θ, z^0) for all pairs (θ, z^0) such that $\theta \geq \theta_0$ and (θ_0, z^0) are as specified above ; moreover, for each $\theta > \theta_0$, any optimal solution $(\widehat{\lambda}, \widehat{u}^{\pm}, \widehat{v})$ of the DLP(θ, z^0) satisfies

$$\boxed{(z^0)^T \widehat{u}^+ + (\,\mathbf{1} - z^0\,)^T \widehat{v} = 0}$$

Removing the parameter θ, continued

Thus, the limiting dual problem for large θ can be expressed:

$$D(z) \quad \begin{array}{ll} \underset{(\lambda, u^{\pm}, v)}{\text{maximize}} & f^T\lambda + q^T(\, u^+ - u^- \,) \\ \text{subject to} & A^T\lambda - N^T(\, u^+ - u^- \,) \;=\; c \\ & B^T\lambda - M^T(\, u^+ - u^- \,) - v \;\le\; d \\ & z^T u^+ + (\, \mathbf{1} - z\,)^T v = 0 \\ \text{and} & (\, \lambda, u^{\pm}, v \,) \;\ge\; 0, \end{array}$$

If $z_i = 1$ then $u_i^+ = 0$.
If $z_i = 0$ then $v_i = 0$.

Outline

1. Introduction and Motivation
2. MIP formulation of LPCCs
3. Global solution framework
    - Master Problem
    - Cuts for the Master Problem
4. Refinements
    - Cut Sparsification
5. Computational Results
    - Feasible LPCCs
    - Unbounded LPCCs
    - Infeasible LPCCs
    - Box-constrained quadratic programs
6. Conclusions

The Master Problem

In order to find the best choice for z and to resolve LPCC, use a logical Benders decomposition method (Hooker; see also Codato and Fischetti).

Initially every binary z is feasible. Satisfiability cuts are added to restrict z based on the solution of the dual problem $D(z)$.

The Master Problem is a Satisfiability Problem.

The algorithm

Outline:

1. Initialize the Master Problem with all binary z feasible.
2. If the Master Problem is infeasible, STOP with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, STOP.
6. Update the Master Problem and return to Step 2.

The algorithm

Outline:

1. Initialize the Master Problem with all binary z feasible.
2. If the Master Problem is infeasible, STOP with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, STOP.
6. Update the Master Problem and return to Step 2.

The algorithm

Outline:

1. Initialize the Master Problem with all binary z feasible.
2. If the Master Problem is infeasible, STOP with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, STOP.
6. Update the Master Problem and return to Step 2.

The algorithm

Outline:

1. Initialize the Master Problem with all binary z feasible.
2. If the Master Problem is infeasible, STOP with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, STOP.
6. Update the Master Problem and return to Step 2.

The algorithm

Outline:

1. Initialize the Master Problem with all binary z feasible.
2. If the Master Problem is infeasible, STOP with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, STOP.
6. Update the Master Problem and return to Step 2.

The algorithm

Outline:

1. Initialize the Master Problem with all binary z feasible.
2. If the Master Problem is infeasible, STOP with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, STOP.
6. Update the Master Problem and return to Step 2.

Implications of solving $D(\bar{z})$: $D(\bar{z})$ finite

If $D(\bar{z})$ is feasible with finite optimal value $\phi(\bar{z})$ then this value gives the optimal value on the corresponding piece of LPCC.

Thus, we obtain an upper bound on the optimal value of LPCC and can restrict attention in the Master Problem to better pieces.

If the optimal solution to $D(\bar{z})$ is feasible in $D(z)$ for some other z then the value of LPCC on the piece corresponding to z must also be at least $\phi(\bar{z})$.

So use a point cut to remove all such z from the Master Problem, based on the optimal solution $(\bar{\lambda}, \bar{u}^{\pm}, \bar{v})$ to $D(\bar{z})$:

$$\sum_{i:\bar{u}_i^+>0} z_i + \sum_{i:\bar{v}_i>0} (1 - z_i) \geq 1$$

This logical Benders cut will force at least one of these components of u^+ or v to be zero in all future subproblems.

Implications of solving $D(\bar{z})$: $D(\bar{z})$ unbounded

If $D(\bar{z})$ is unbounded then the corresponding piece of LPCC is infeasible.

Have a ray for $D(\bar{z})$.

Cut off all z in the Master Problem for which this ray $(\bar{\lambda}, \bar{u}^{\pm}, \bar{v})$ is feasible in $D(z)$, using a ray cut:

$$\sum_{i:\bar{u}_i^+>0} z_i + \sum_{i:\bar{v}_i>0} (1 - z_i) \geq 1$$

Implications of solving $D(\bar{z})$: $D(\bar{z})$ infeasible

If $D(\bar{z})$ is infeasible then the corresponding piece of LPCC is either infeasible or unbounded.

Solve a homogenized version of $D(\bar{z})$ to determine the case:

$$D_0(\bar{z}) \qquad \begin{array}{ll} \underset{(\lambda, u^{\pm}, v)}{\text{maximize}} & f^T\lambda + q^T(u^+ - u^-) \\ \text{subject to} & A^T\lambda - N^T(u^+ - u^-) = 0 \\ & B^T\lambda - M^T(u^+ - u^-) - v \leq 0 \\ & \bar{z}^T u^+ + (\mathbf{1} - \bar{z})^T v = 0 \\ \text{and} & (\lambda, u^{\pm}, v) \geq 0, \end{array}$$

If $D_0(\bar{z})$ is unbounded then the corresponding primal problem is infeasible. Thus, the corresponding piece of LPCC is infeasible, so we can again add a ray cut.

If $D_0(\bar{z})$ has optimal value 0 then LPCC is unbounded.

Outline

1. Introduction and Motivation
2. MIP formulation of LPCCs
3. Global solution framework
    - Master Problem
    - Cuts for the Master Problem
4. **Refinements**
    - Cut Sparsification
5. Computational Results
    - Feasible LPCCs
    - Unbounded LPCCs
    - Infeasible LPCCs
    - Box-constrained quadratic programs
6. Conclusions

Some useful auxiliary steps

Simple cuts (Audet, Savard, and Zghal; JOTA, in print)
to tighten the joint constraints $Ax + By \geq f$ using the complementarity resrictions

LPCC feasibility recovery
to improve LPCC upper bounds if possible

Cut sparsification
to tighten up the cuts added to the Master Problem

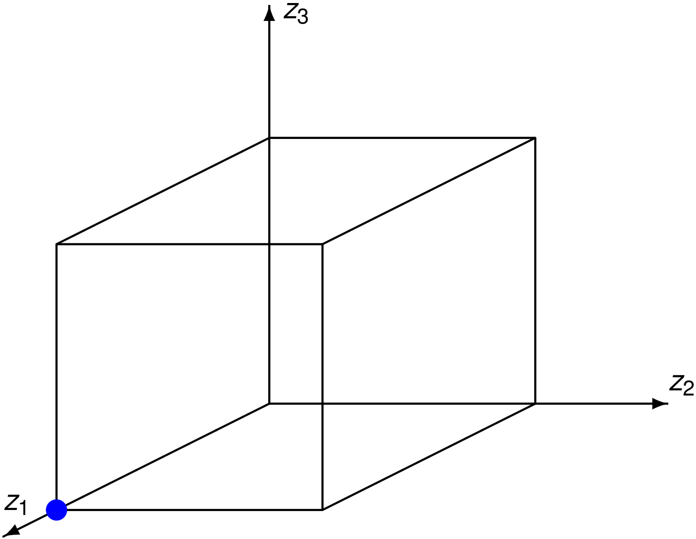
Cut Sparsification

The fewer variables included in a point cut or ray cut, the tighter the satisfiability constraint.

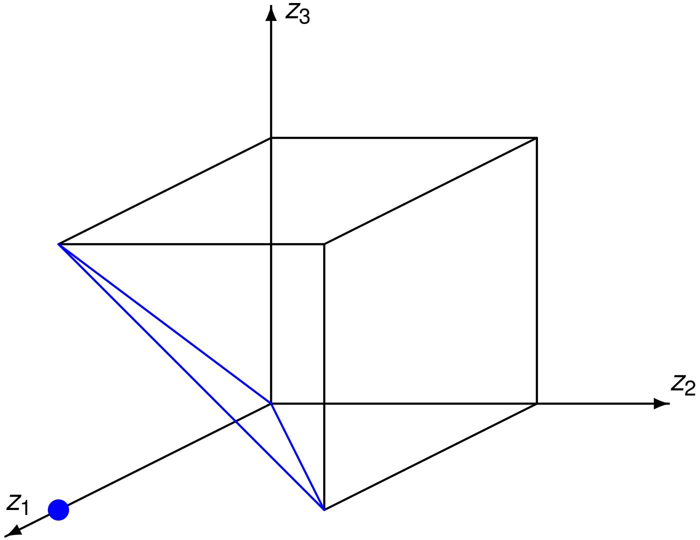
We use various heuristic procedures to try to sparsify the cut. These heuristics require the solution of linear programs.

In the case of a ray cut, we are looking for an irreducible infeasible set (IIS) of constraints for the primal problem $P(\bar{z})$ that is dual to $D(\bar{z})$.

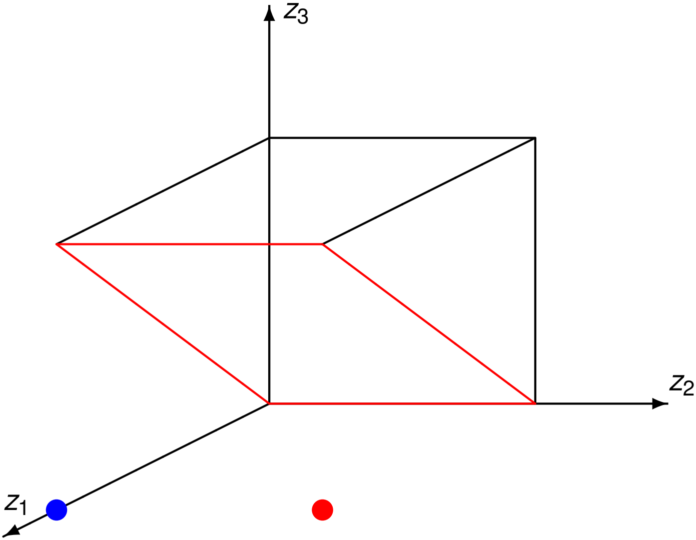
Initial feasible region. Take $\bar{z} = (1, 0, 0)$.



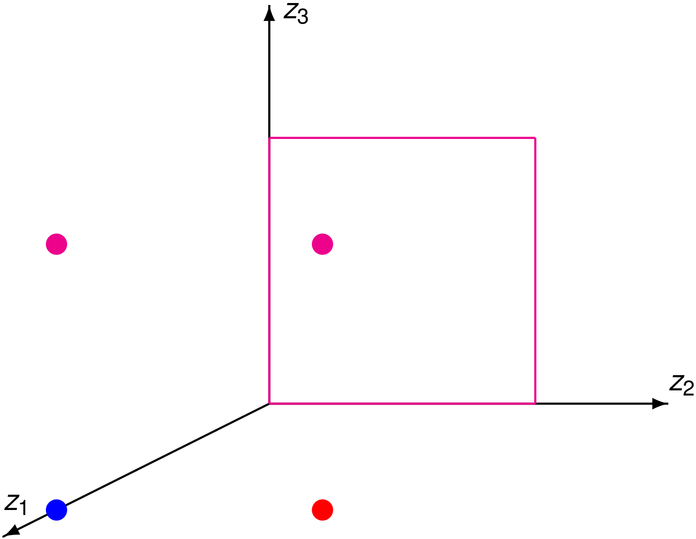
Add cut $(1 - z_1) + z_2 + z_3 \geq 1$ to cut off $\bar{z} = (1, 0, 0)$



Sparsify to $(1 - z_1) + z_3 \geq 1$, cuts off $z = (1, 1, 0)$



Sparsify further to $(1 - z_1) \geq 1$, cuts off 2 more pts



Outline

1. Introduction and Motivation
2. MIP formulation of LPCCs
3. Global solution framework
    - Master Problem
    - Cuts for the Master Problem
4. Refinements
    - Cut Sparsification
5. Computational Results
    - Feasible LPCCs
    - Unbounded LPCCs
    - Infeasible LPCCs
    - Box-constrained quadratic programs
6. Conclusions

Feasible LPCCs with $B = 0$, $A \in \Re^{200\times 300}$, and 300 complementarities

Prob	LPCC$_{\text{min}}$	FILTER	KNITRO	LPs	IPs
1	2478.2254	2478.2256	2478.2264	125	1
2	3270.1842	3280.1865	3270.1844	4071	62
3	3660.5407	3660.5412	3660.5412	350	2
4	3176.4109	3176.4108	3176.4115	1249	15
5	2959.9498	2959.9495	2959.9529	5	1
6	2672.5706	2684.5288	2672.5710	4511	70
7	2617.2640	2617.2638	2617.2673	0	0
8	2771.2374	2771.2372	2771.2379	26	1
9	2847.6923	2847.6926	2847.6929	319	2
10	3230.9893	3230.9896	3230.9897	1569	16

Feasible general LPCCs with $B \neq 0$, $A \in \Re^{55\times 50}$, and 50 complementarities.

Prob	LPCC$_{\text{min}}$	FILTER	KNITRO	LPs	IPs
1	29.0501	29.0501	30.0155	21	2
2	37.5509	37.5509	37.5510	229	9
3	37.0022	38.3216	38.7521	4842	696
4	34.2228	34.6057	34.2398	102	7
5	22.2835	22.2945	22.2837	209	24
6	30.0829	30.0829	30.0830	108	13
7	38.0405	38.0419	38.0419	92	7
8	22.3969	22.7453	22.4164	187	21
9	40.3380	44.7872	44.3173	321	14
10	41.3957	41.5810	41.5810	190	19

Unbounded LPCCs with 50 complementarities

Prob	# iters	# cuts	# LPs
1	50	47	195
2	6	4	14
3	1081	828	2604
4	166	144	424
5	436	305	991
6	18	17	54
7	3	4	11
8	426	356	1191
9	9	9	26
10	4	3	11

iters = number of Master Problem iterations

cuts = number of satisfiability constraints in Master Problem at termination

LPs = number of LPs solved, excluding the pre-processing step

Infeasible LPCCs with 50 complementarities

Prob	# iters	# cuts	# LPs
1	14	14	28
2	2	2	4
3	38	38	76
4	7	7	14
5	47	49	100
6	48	48	96
7	20	20	40
8	13	13	26
9	50	50	100
10	6	6	12

iters = number of Master Problem iterations
cuts = number of satisfiability constraints in Master Problem at termination
LPs = number of LPs solved, excluding the pre-processing step

LPCC formulation of QP

The optimal solution to the box-constrained quadratic program

$$\begin{array}{ll} \min & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & 0 \leq x \leq \mathbf{1} \end{array}$$

can be found by solving the LPCC:

$$\begin{array}{ll} \min & c^T x - \mathbf{1}^T y \\ \text{subject to} & 0 \leq x \perp c + Qx + y \geq 0 \\ & 0 \leq \mathbf{1} - x \perp y \geq 0 \end{array}$$

Preliminary computational results

Problems typically solved in a few seconds:
$n = 50$ with density 25%
$n = 75$ with density 10%
$n = 100$ with density 5%

Problems where some instances cannot currently be solved:
$n = 100$ with density 10%

Problems where the current implementation typically has great difficulties:
$n = 75$ with density 25%

Outline

1. Introduction and Motivation
2. MIP formulation of LPCCs
3. Global solution framework
   - Master Problem
   - Cuts for the Master Problem
4. Refinements
   - Cut Sparsification
5. Computational Results
   - Feasible LPCCs
   - Unbounded LPCCs
   - Infeasible LPCCs
   - Box-constrained quadratic programs
6. Conclusions

Conclusions

The logical Benders decomposition method can successfully find the global solution to large feasible LPCC instances, often finding better solutions than NLP methods which determine local minimizers.

The method successfully identifies infeasible or unbounded LPCC instances.

The method can be used to solve bounded quadratic programming problems. Extension: We can also formulate a general QP as an LPCC, which can even identify unbounded QPs.