Conic Cutting Surface Algorithms

John E. Mitchell   Vasile L. Basescu

Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, NY 12180 USA

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Conic Cuts

- Convex feasibility problem
- Cutting planes
- Restarting
- Convergence

Selective Orthonormalization

- Motivation
- Modifying a single cut
- Modifying multiple cuts

Conclusions

Abstract

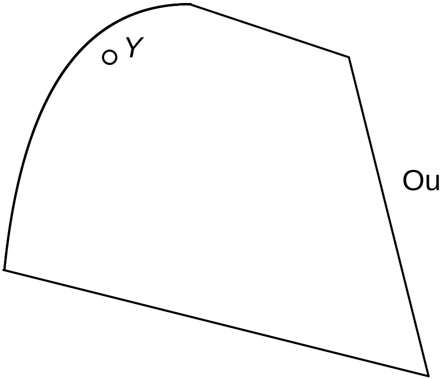
The problem of finding a feasible point in a fully dimensional set in a finite dimensional Hilbert space is analyzed. An analytic center cutting surface algorithm is developed that adds conic cuts. The algorithm generalizes similar LP, SDP, and SOCP approaches. It is shown that the algorithm is fully polynomial, with the complexity dependent on a condition number of the cuts. The algorithm is refined by modifying the cuts, which allows the derivation of a complexity result that does not depend on this condition number.

Convex feasibility problem

Given a set Y, find a point $y \in Y$ or determine that Y is empty.
Assumptions:

- Y is a convex, bounded set contained in $\mathbb{R}^m$, containing a ball $B(., \epsilon)$ of radius ϵ.
- If $\bar{y} \notin Y$, a separation oracle returns a conic inequality $G^* y + s = h$, $s \in K_0$ satisfied by all $y \in Y$ and violated by $\bar{y}$. K_0 is a full-dimensional self-scaled cone in R^p.
- Assume the cone K_0 has a self-concordant barrier function $f_0(K_0)$.

Convex feasibility problem



Outer approximation of Y

Conic relaxation

The feasibility problem is approximated by a **conic program**:

$$\begin{array}{lrcrcl} \max & 0 & & & & \\ \text{subject to} & A^*y & + & s & = & c \\ & & & s & \in & K \end{array} \qquad (CD)$$

where K is a full-dimensional self-scaled cone.

Note that K may be a product of smaller cones.
Eg, $K = R^n_+$, $K =$ SDP cone, $K =$ product of SOCP cones, ...

Duality

Call (CD) the dual problem. The corresponding primal problem is:

$$\begin{array}{llcl} \min & <c, x> & & \\ \text{subject to} & Ax & = & 0 \\ & x & \in & K \end{array} \qquad (CP)$$

Note that we assume that K is self-dual.

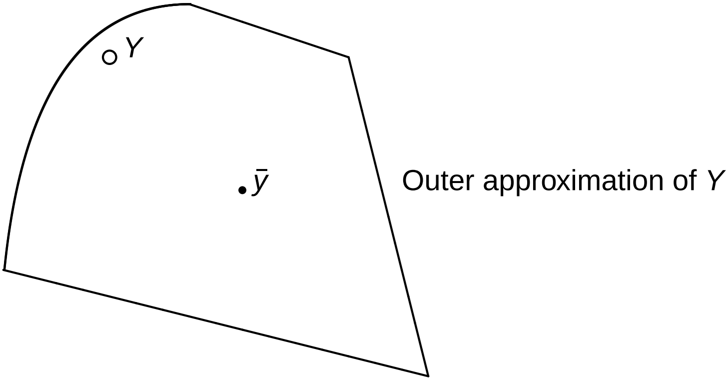
Cutting planes

- We find an **approximate analytic center** $(\bar{x}, \bar{y}, \bar{s})$ for (CP) and (CD).
- If $\bar{y} \in Y$, **DONE**.
- Otherwise, the oracle returns a **cut** violated by $\bar{y}$ and satisfied by all $y \in Y$:

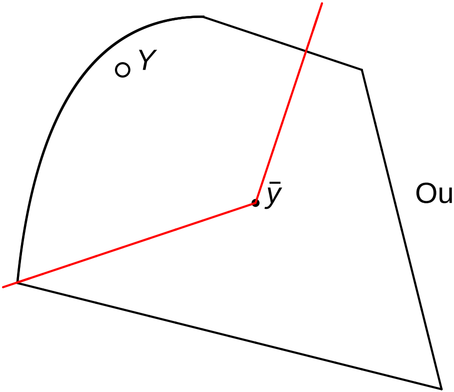
$$\begin{array}{rcrcl} G^*y & + & s & = & h \\ & & s & \in & K_0 \end{array}$$

- We assume the cut is shifted to be **central**, so $h = G^*\bar{y}$.
- The modified primal problem can still be restarted effectively, even if the cut is deep.

Find approximate analytic center $\bar{y}$

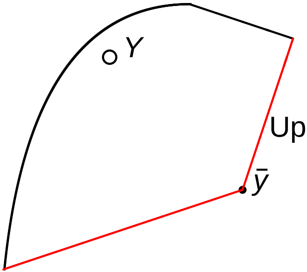


Add a conic cut at $\bar{y}$



Outer approximation of Y

Update outer approximation



Updated outer approximation of Y

Modifying (CP) and (CD)

$$\begin{array}{llcccl} \text{max} & 0 & & & & \\ \text{subject to} & A^*y & + & s & = & c \\ & G^*y & + & s_0 & = & h \\ & s \in K, & & s_0 & \in & K_0 \end{array} \qquad (\overline{CD})$$

$$\begin{array}{llcccl} \text{min} & <c,x> & + & <h,x_0> & & \\ \text{subject to} & Ax & + & Gx_0 & = & 0 \\ & x \in K, & & x_0 & \in & K_0 \end{array} \qquad (\overline{CP})$$

With $x_0 = 0$ and $s_0 = 0$, the current point $(\bar{x}, \bar{y}, \bar{s})$ is on the boundary of the feasible regions of $(\overline{CD})$ and $(\overline{CP})$.

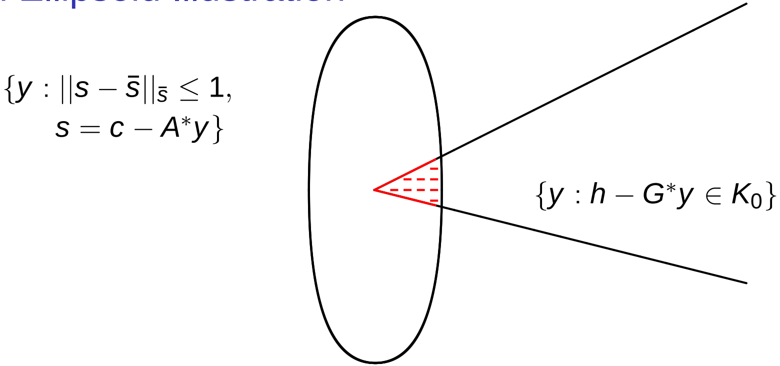
Dikin Ellipsoids

- Want to find s_0 and x_0 in the **interior** of K_0. Assume the cone K_0 has a self-concordant barrier function $f_0(K_0)$, with conjugate function $f_0^*(K_0)$.
- Try to **minimize this barrier function over s_0 (or x_0) subject to keeping y, s (or x) in an appropriately defined Dikin ellipsoid**:

$$\begin{array}{lrcrcl} \text{min} & & & f_0^*(s_0) & & \\ \text{subject to} & A^*y & + & s & = & c \\ & G^*y & + & s_0 & = & h \\ & & & ||s - \bar{s}||_{\bar{s}} & \leq & 1 \\ & & & s_0 & \in & K_0 \end{array}$$

Here $||v||_u := ||H(u)^{1/2}v||$, with $H(u)$ the Hessian of f.

Dikin Ellipsoid Illustration



$$\{y : ||s - \bar{s}||_{\bar{s}} \leq 1, \quad s = c - A^* y\}$$

$$\{y : h - G^* y \in K_0\}$$

Due to Goffin and Vial for LP, generalized to SDP and SOCP by Oskoorouchi and Goffin, and to more general problems by Basescu and Mitchell. Generalizes the single cut approach of Mitchell and Todd.

Finding a new approximate analytic center

- The **primal-dual potential function** is

  $$\Phi_{PD} \;=\; <x, s> + f(x) + f^*(s).$$

  It is minimized with value 0 at the analytic center.
- Let ϑ_f denote the **complexity value of f**, in the terminology of Renegar's text.

  LP: $f(x) = -\sum_{i=1}^n \ln(x_i)$, $\vartheta_f = n$.
  SDP: $f(X) = -\det(X)$, $\vartheta_f = n$.
  SOCP: $f(\xi, x) = -0.5\ln(\xi^2 - \sum_{i=1}^n x_i^2)$, $\vartheta_f = 2$.

Convergence

- **Local convergence:**
  Get convergence to new approximate analytic center in $O(\vartheta_{f_0} \ln \vartheta_{f_0})$ steps.
- **Global convergence:**
  Polynomial in m, the required tolerance ϵ,
  and a *condition number* based on the added constraints.

The need for a condition number

- The proof of global convergence examines the dual potential function $f^*(c - A^*y)$.
- Upper bound from assumption of an ϵ-ball:

  If $c - A^*(y + \epsilon u) \succeq_K 0$ for any unit vector u then $f_K^*(c - A^*y) \leq f_K^*(\epsilon A^*u) = f_K^*(A^*u) - \vartheta_{f_K} \ln \epsilon$.
- Define condition number μ_K so that $\ln \mu_K := \inf\{f_K^*(A^*u) : ||u|| = 1\}$.
- Need an upper bound if $Y = \emptyset$, so can stop with a guarantee that the set is empty.

A second order cone with a bad condition number

- Take a scalar y and define

$$c = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } A^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- For any $u \in \mathbb{R}$ we get A^*u on the boundary of the second order cone.
- The constraint $c - A^*y \succeq_K 0$ is equivalent to the linear constraint $y \leq 1$.

Selective Orthonormalization

Selective Orthonormalization is a method for modifying the added cuts so that

- It is easy to restart
- The complexity does not depend on a condition number
- Drawback: It may weaken the cuts.

Modifying a single cut

Given a central constraint $c - A^*y \succeq_K 0$:

- Want a direction d such that $A^*d \prec_K 0$.
- Let e be in interior of K. Can assume $Ae \neq 0$ (else no such d exists).
- Let $\bar{A} := A + \lambda Aee^*$ with $\lambda \geq 0$.
- Then $\bar{A}^*d \preceq_K A^*d$ for all d satisfying $A^*d \preceq_K 0$.

  So $c - \bar{A}^*y \succeq_K 0$ is a valid constraint.

Example

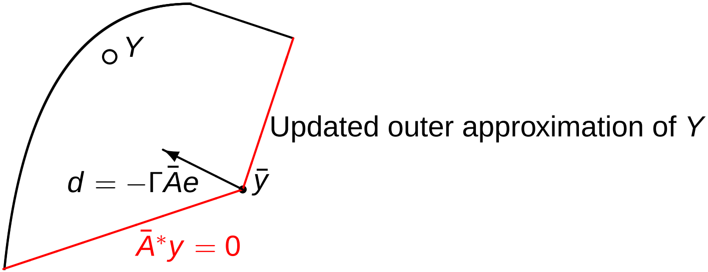
- Use earlier second order cone example with $A = [1, 1]$.
- Take $e^T = [1, 0]$. Then

  $\bar{A} = [1 + \lambda, 1]$ and

  $\bar{A}^*(\bar{A}e) = [(1 + \lambda)^2, 1 + \lambda]^T$, which is in the interior of K.

Choosing λ for strict feasibility

Want to move in the direction $d = -\Gamma\bar{A}e$, where $\Gamma := (AH(\bar{s})A^*)^{-1}$. Need $\bar{A}^*d \prec_K 0$.



Chossing λ to control the condition number

- To control the condition number, update $\bar{A}$ so that $\bar{A}^*\bar{A}e \succeq_K \omega e$, where ω is chosen to be a scalar between 0 and 1.
- Then $\ln \mu_K \leq -\vartheta_{f_K}(1 + \ln(\omega))$

  and $f_K^*(c - A^*y) \leq -\vartheta_{f_K}(1 + \ln(\epsilon\omega))$ for $B(y, \epsilon) \subseteq K$.

  (Assume A and e each have norm 1, and $f_K(e) = 0$, so $f_K^*(e) = -\vartheta_{f_K}$.)

Technicalities of picking λ

- If $\bar{A}^*\bar{A}e \not\succeq_K 0$, set $\bar{\lambda} := \min\{\lambda : \lambda e + \frac{1}{||\bar{A}e||}\bar{A}^*\bar{A}e \succeq_K 0\}$. Update $\bar{A} \leftarrow \bar{A} + \bar{\lambda}\bar{A}ee^T$ and renormalize so that $||\bar{A}|| = 1$.
- If $\bar{A}^*\bar{A}e \not\succeq_K \omega e$, update $\bar{A} \leftarrow (1-\sqrt{\omega})\bar{A} + \frac{\sqrt{\omega}}{||\bar{A}e||}\bar{A}ee^T$. Renormalize so that $||\bar{A}|| = 1$.
- Let $\eta = \sqrt{e^*\bar{A}^*\Gamma\bar{A}e}$. If $\bar{A}^*\Gamma\bar{A}e \not\succeq_K \nu\eta^2 e$, update $\bar{A} \leftarrow (1-\nu)\bar{A} + \nu\bar{A}ee^T$. Renormalize so that $||\bar{A}|| = 1$.

Multiple cuts

- Assume multiple central cuts are added simultaneously.
- Can extend Selective Orthonormalization to ensure that can still restart easily.
- Need to modify using terms of the form

  $$\bar{A}_p \leftarrow \bar{A}_p + \lambda \bar{A}_q e_q e_p^*$$

  to ensure $\bar{A}_p^* \Gamma \bar{A}_q e_q \succeq_K 0$.
- Use direction

  $$d = -\sum_{p=1}^{l} \frac{1}{\eta_p} \Gamma \bar{A}_p e_p$$

Conclusions

- Given a convex feasibility problem with an oracle that returns conic cuts, the cutting surface method can determine feasibility in fully polynomial time.
- The complexity does not depend on a condition number if selective orthonormalization is used.
- Open question: can the condition number be eliminated from the complexity without weakening the cuts?

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