An LPCC Approach to Indefinite Quadratic Programs

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Outline

Linear Programs with Complementarity Constraints

Find $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ to globally solve the linear program with complementarity constraints (LPCC):

$$\begin{array}{ll} \underset{(x,y,w)}{\text{minimize}} & c^T x + d^T y \\ \text{subject to} & Ax + By \geq f \\ \text{and} & 0 \leq y \perp w := q + Nx + My \geq 0, \end{array}$$

Fundamental importance

Novel paradigms in mathematical programming:

- hierarchical optimization:

some of the variables have to be optimal solutions to another problem.

- inverse optimization

find appropriate parameters so that a given solution really solves an optimization problem

Key formulations for:

- global resolution of nonconvex quadratic programs
- piecewise linear problems
- quantile minimization
- network design

Outline

Nonconvex quadratic programming problems

Quadratic program:

$$\begin{array}{lll} \min_x & c^T x + \frac{1}{2} x^T Q x & \\ \text{s.t.} & Ax \leq b & \end{array}$$

LPCC formulation:

$$\begin{array}{ll} \min_{x,\xi} & \frac{1}{2} c^T x - \frac{1}{2} b^T \xi \\ \text{s.t.} & c + Qx + A^T \xi = 0 \\ & 0 \leq \xi \perp b - Ax \geq 0 \end{array}$$

The LPCC formulation finds the best KKT point.

But: it doesn't identify unboundedness.

Paul Tseng: A finite procedure based on examining extreme points and rays can be used to determine unboundedness (private communication).

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Conditions for unboundedness

Eaves (1971, *Management Science*):

Theorem

The QP is unbounded if and only if there exists a feasible point $\bar{x}$ and a direction d in the recession cone satisfying $(c + Q\bar{x})^T d < 0$.

Corollary

A feasible QP is unbounded if and only if either Q is not copositive on the recession cone or there exists a direction d in the recession cone satisfying $d^T Q d = 0$ and there exists a feasible point $\bar{x}$ satisfying $(c + Q\bar{x})^T d < 0$.

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Constructing a ray QP and LPCC

Assume WLOG recession cone contained in the nonnegative orthant.

Can set up QP to determine if Q is copositive on recession cone:

Quadratic program:

$$\begin{array}{lrcl} \min_d & \frac{1}{2} d^T Q d & & \\ \text{s.t.} & Ad & \leq & 0 \\ & \mathbf{1}_n^T d & = & 1 \end{array}$$

LPCC ray formulation:

$$\begin{array}{ll} \min_{d,\lambda,s} & -\frac{1}{2} s \\ \text{s.t.} & Qd + A^T\lambda + s\mathbf{1}_n = 0 \\ & 0 \leq \lambda \perp -Ad \geq 0 \\ & s \text{ free}, \ \mathbf{1}_n^T d = 1 \end{array}$$

Optimal value $\geq 0 \iff Q$ is copositive on recession cone.

Optimal value $> 0 \iff Q$ is strictly copositive on recession cone.

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Setting up a parametrized QP

Can put a parametrized bound on sum of components.

Parametrized QP:

$$
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\min_x & c^T x + \frac{1}{2} x^T Q x & & \\
\text{s.t.} & Ax & \leq & b \\
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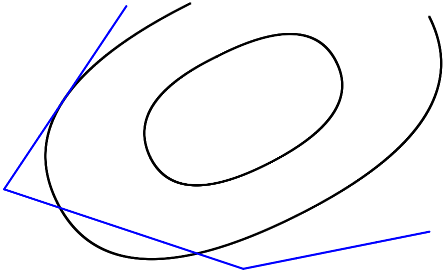

Case with bounded optimal value

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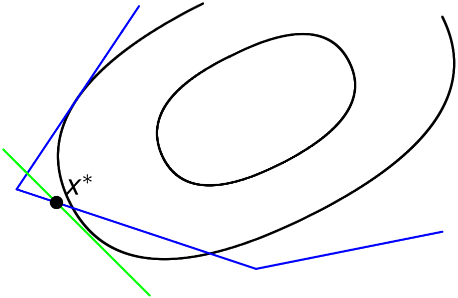

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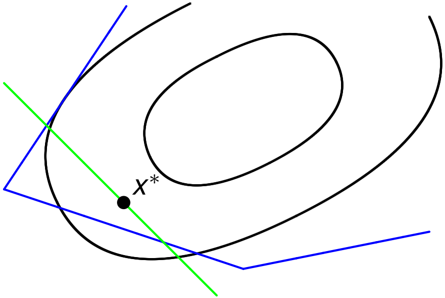

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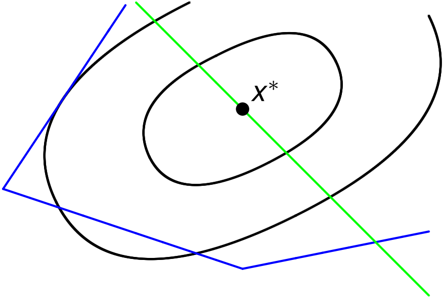

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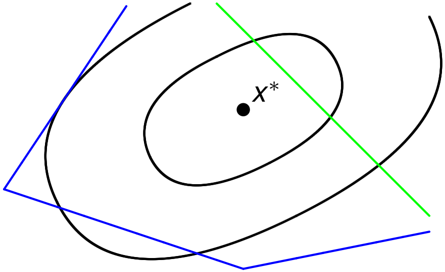

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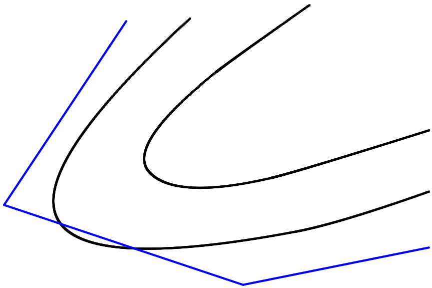

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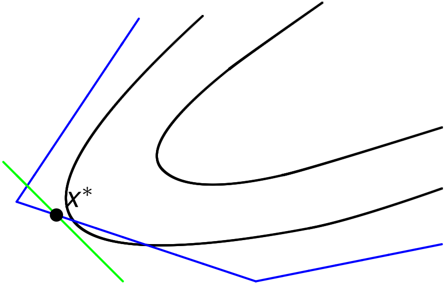

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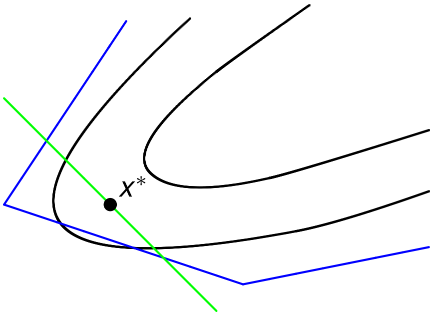

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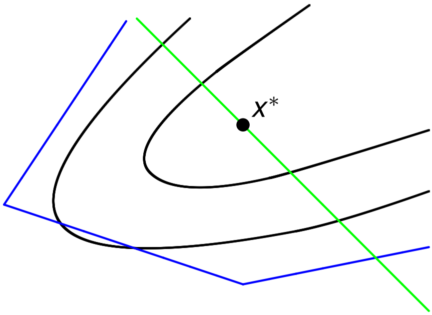

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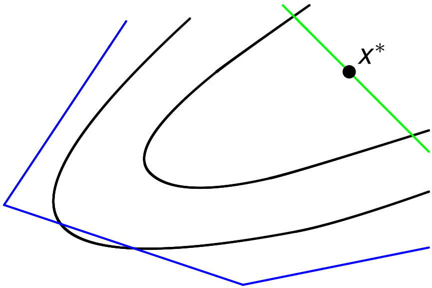

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The parametrized QP has a bounded feasible region for any R, so it suffices to search for a KKT point.

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$$\begin{array}{ll} \min_{x,\xi,t} & \frac{1}{2} c^T x - \frac{1}{2} b^T \xi - \frac{1}{2} R t \\ \text{s.t.} & c + Qx + A^T \xi + \mathbf{1}_n t = 0 \\ & 0 \leq \xi \perp b - Ax \geq 0 \\ & 0 \leq t \perp R - \mathbf{1}_n^T x \geq 0 \end{array}$$

Finite optimal value: get $t = 0$ for sufficiently large R.

Unbounded optimal value: get $t > 0$ as $R \to \infty$ (at least on an infinite subsequence).

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LPCC to determine unboundedness

$$\max \quad t$$

$$\text{s.t.} \quad 0 = c + Qx + A^T\xi + t\,\mathbf{1}_n \quad \text{Lagrangian equation of truncated QP}$$

$$0 \leq \xi \perp b - Ax \geq 0 \quad \text{standard complementarity}$$

(drop the condition $R - \mathbf{1}_n^T x \geq 0$).

LPCC to determine unboundedness

$$\max \quad t$$

s.t.	$0 = c + Qx + A^T\xi + t\,\mathbf{1}_n$	Lagrangian equation of truncated QP
	$0 = Qd + A^T\lambda + s\,\mathbf{1}_n$	from the ray KKT system
	$0 \leq \xi \perp b - Ax \geq 0$	standard complementarity
	$0 \leq \lambda \perp -Ad \geq 0$	standard ray complementarity
	$0 \leq \xi \perp -Ad \geq 0$	connecting KKT multiplier with ray
	$0 \leq s, \quad \mathbf{1}_n^T d = 1$	ensuring nonzero ray.

(replace it by conditions from the recession cone).

LPCC to determine unboundedness

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Theorem

QP unbounded $\iff$ *LPCC has feasible solution with* $t > 0$.

Sketch of proof, part I

LPCC solution with $t > 0$ implies QP unbounded:

$$
\begin{aligned}
0 &= c + Qx + A^T\xi + t\,\mathbf{1}_n \\
0 &= Qd + A^T\lambda + s\,\mathbf{1}_n \\
0 &\leq \xi \perp b - Ax \geq 0 \\
0 &\leq \lambda \perp -Ad \geq 0 \\
0 &\leq \xi \perp -Ad \geq 0 \\
0 &\leq s, \quad \mathbf{1}_n^T d = 1,
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
0 > -t &= -t\mathbf{1}_n^T d \\
&= (c + Qx + A^T\xi)^T d \\
&= (c + Qx)^T d
\end{aligned}
$$

and x is feasible, d is in the recession cone.

Sketch of proof, part II

QP unbounded implies LPCC solution with $t > 0$:

Parametrized LPCC:

$$\begin{array}{ll} \min_{x,\xi,t} & \frac{1}{2}c^T x - \frac{1}{2}b^T\xi - \frac{1}{2}Rt \\ \text{s.t.} & c + Qx + A^T\xi + \mathbf{1}_n t = 0 \\ & 0 \leq \xi \perp b - Ax \geq 0 \\ & 0 \leq t \perp R - \mathbf{1}_n^T x \geq 0 \end{array}$$

As $R \to \infty$, LPCC value $\to -\infty$.

Pick an assignment of the complementarity relationships that occurs infinitely often.

Decompose into $\bar{x} + td$ with $\mathbf{1}_n^T d = 1$.

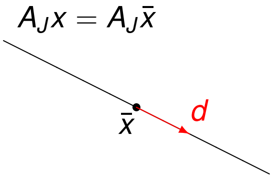
Then look at the limits.

(Proof uses fact that (QP) only attains finitely many values on its KKT points (Luo and Tseng, 1992, *SIOPT*).)

Outline

Second order cuts

Exploit the second order KKT conditions:



Assume Q not pd on nullspace of A_J, so

$\exists d \neq 0$ in $\mathcal{N}(A_J)$ with $d^T Q d \leq 0$. Then

(i) If $d^T Q d < 0$ or $(c + Q\bar{x})^T d \neq 0$:
Either d or $-d$ is an improving direction.

(ii) If Q psd on $\mathcal{N}(A_J)$ and $(c + Q\bar{x})^T d = 0$:
Either d or $-d$ leads to a feasible point with more active constraints and same objective function value.

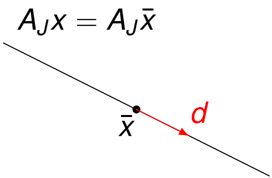
Can rule out points with active set $\subseteq J$ where Q is not pd on $\mathcal{N}(A_J)$:

Theorem

One of the constraints in $\bar{J}$ must be active.

Second order cuts

Exploit the second order KKT conditions:



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Eg: Box constrained QPs

Nonconvex QP with simple constraints:

$$\begin{array}{ll} \min_x & c^T x + \frac{1}{2} x^T Q x \\ \text{s.t.} & 0 \leq x \leq \mathbf{1}_n \end{array}$$

Hansen, Jaumard et al (1993, *NRLQ*): if $Q_{ii} \leq 0$ then there is an optimal solution with $x_i = 0$ or 1.

Our result implies:
If $Q_{\bar{J}}$ is a principal minor of Q that is not positive semidefinite then $x_j = 0$ or 1 for at least one component $j \in \bar{J}$.

Computationally speaking

Experiments with box-constrained QPs, generated by Vandenbussche and Nemhauser (2005, *Math Progg*) and Burer and Vandenbussche (2009, *COAP*).

Preprocess: Add all constraints corresponding to non-psd principal minors up to size $k \times k$.

Improves runtimes for $k = 1$ and $k = 2$.

Not so helpful for $k \geq 3$:
the additional constraints are typically not strong enough to justify the computational overhead of generating them and working with them.

Outline

Polynomially solvable classes of QPs

Hager, Pardalos et al (1991, *JOTA*): Look at the QP

$$\begin{array}{ll} \min_x & c^T x + \frac{1}{2} x^T Q x \\ \text{s.t.} & x \geq 0 \end{array}$$

If number of negative eigenvalues of Q is no greater than fixed k then the best KKT point can be found in polynomial time.

Pardalos, Vavasis (1991, *JOGO*):
General QP with one negative eigenvalue is NP-hard.
(Construction has only one nonzero eigenvalue.)

Polynomially solvable classes of QPs

Theorem

If Q is not positive definite on $\mathcal{N}(A_J)$ then at least one of the constraints in $\bar{J}$ must be active.

Theorem

Let A be $(n+k) \times n$. Let Q have no more than p positive eigenvalues. Let k and p both be fixed. Assume the feasible region of QP is bounded. Then the global optimum can be found in polynomial time.

Corollary

Testing the copositivity of a matrix with a bounded number of positive eigenvalues on a polyhedral cone $\subseteq \mathbb{R}^n_+$ defined by a bounded number of linear inequalities can be accomplished in polynomial time.

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Conclusions

A single LPCC can be used to determine whether a quadratic program is unbounded.

LPCC formulations of quadratic programs can be tightened by using cuts that exploit the second order KKT conditions.

Certain classes of nonconvex quadratic programs can be solved in polynomial time.