Branch-and-Price-and-Cut on Clique Partition Problem with Minimum Size Requirement

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Problem Definition

Clique Partition Problem with MINimum Clique Size CPPMIN

- Given an **integer** $S > 0$ and a **complete graph** $G = (V, E)$ with weights c_e on each edge e.
- Partition the vertices into subsets with **at least S vertices**, so as to minimize the total weight of edges having both endpoints in the same set.

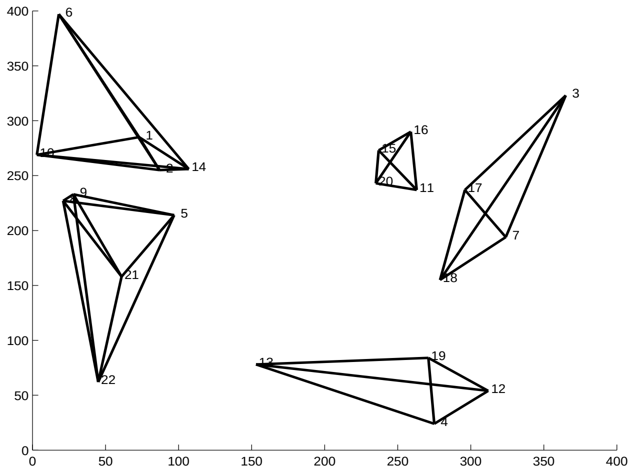

Figure 1: One solution for a problem of size $n = 22$, $S = 4$

IP Formulation

For any $C \subseteq V, |C| \geq S$, define a binary variable x_C,

$$x_C = \begin{cases} 1 & \text{if cluster C is used in the solution of CPPMIN} \\ 0 & \text{otherwise} \end{cases}$$

Master Problem

$$\begin{aligned} \min \quad & \sum_{C} w_C x_C \\ \text{s.t.} \quad & \sum_{C:v \in C} x_C = 1 \quad \forall v \in V \\ & x_C \geq 0 \\ & C \in 2^{|V|}, |C| \geq S \end{aligned}$$

Restricted Master Problem (iteration k): $Q^k \subseteq 2^{|V|}$

$$\begin{aligned} \min \quad & \sum_{C \in Q^k} w_C x_C \\ \text{s.t.} \quad & \sum_{C:v \in C} x_C = 1 \quad \forall v \in V \\ & x_C \geq 0 \\ & C \in Q^k, |C| \geq S \end{aligned}$$

Where $w_C = \sum_{e \in E(C)} w_e$, $2^{|V|} := \{C : C \subseteq V\}$

Pricing problem: $Q^k \rightarrow Q^{k+1}$ – An Example

Divide a graph of 3 nodes into clusters bigger than size 2.

$$\begin{array}{llllllll} \min & w_1x_1 & +w_2x_2 & +w_3x_3 & +w_4x_4 & & & \\ \text{s.t.} & x_1 & +x_2 & & +x_4 & = & 1 & \rightarrow \pi_1 \\ & & +x_2 & +x_3 & +x_4 & = & 1 & \rightarrow \pi_2 \\ & x_1 & & +x_3 & +x_4 & = & 1 & \rightarrow \pi_3 \\ & x_i & & & & \geq & 0 & \rightarrow \alpha_i \end{array} \qquad (1)$$

$$\begin{array}{llllllllll} \max & \pi_1 & +\pi_2 & +\pi_3 & & & & & & \\ \text{s.t.} & \pi_1 & & +\pi_3 & +\alpha_1 & & & & = & w_1 \\ & \pi_1 & +\pi_2 & & & +\alpha_2 & & & = & w_2 \\ & & \pi_2 & +\pi_3 & & & +\alpha_3 & & = & w_3 \\ & \pi_1 & +\pi_2 & +\pi_3 & & & & +\alpha_4 & = & w_4 \\ & & & & \alpha_i & & & & \geq & 0 \quad \rightarrow x_i \end{array} \qquad (2)$$

Only when the reduced cost of x_4, $w_4 - (\bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3) < 0$, do we need to add in cluster $C_4 = \{v_1, v_2, v_3\}$.

Pricing Problem

MINimum Node-Edge-Weighted Cluster Problem MINNEWCP

with minimize size constraints

- Given an **integer** $S > 0$ and a **complete graph** $G = (V, E)$ with weight c_e on each edge e, π_v on each node v.
- Find a subset of the vertices $P \subseteq V$ of **at least S vertices**, so as to minimize $\sum_{e \in E(P)} c_e - \sum_{v \in P} \pi_v$.

Initial Pricing Problem

$$\begin{aligned} \min \quad & \tfrac{1}{2} y^T \mathcal{C} y - \pi^T y \\ \text{s.t.} \quad & \sum_{v=1}^{n} y_v \geq S \\ & y_v \in \{0,1\} \end{aligned}$$

Final Pricing Problem

$$\begin{aligned} \min \quad & \tfrac{1}{2} y^T \mathcal{C} y - \pi^T y \\ \text{s.t.} \quad & \sum_{v=1}^{n} w_v y_v \geq S \\ & \sum_{v \notin T_i} y_v \geq 1 \qquad i = 1.^{.t} \\ & y_v \in \{0,1\} \end{aligned}$$

Linearize by introducing variables $z_{ij} = y_i y_j$, then solve it as an IP.

Problem? LP relaxation of Master problem is not tight

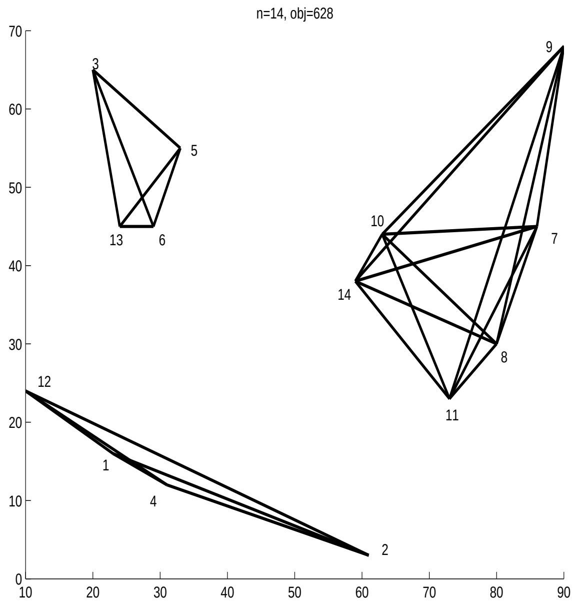

Figure 2: Optimal Solution for a problem of 14 vertices with $S = 4$. $obj = 628$

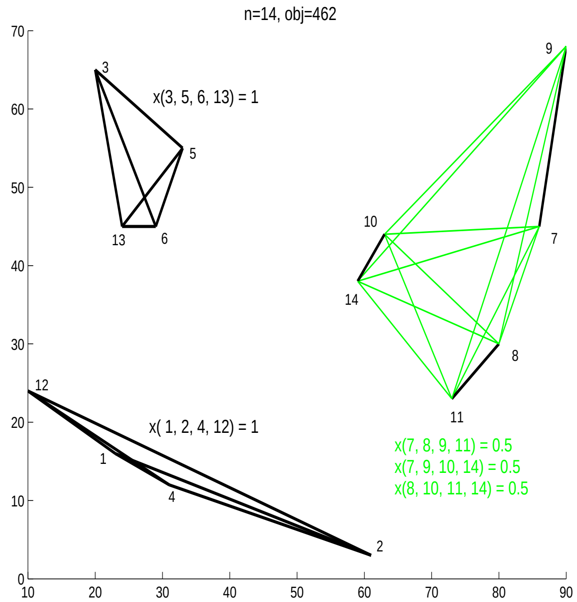

Figure 3: (MLP) Solution for the same problem. $obj = 462$, $\sum_{i:C_i \subseteq \{7,8,9,10,11,14\}} x_i = 1.5 > 1$

Solution? Adding Cutting Planes

For any $T \subseteq V$ where $|T| < kS$.

$$\sum_{i:C_i \subseteq T} x_i \leq k - 1 \tag{3}$$

Combining row generation and column generation is usualy very hard, but possible in this example.

- Check all subsets of T between size S and $2S - 1$.
- Solve the Pricing Problem as an IP with addition constraints.

Other heuristics can be used before calling IP solver to solve the pricing problem.

New Restricted Master Problem and its dual

$$
\begin{array}{llllll}
\min & \sum_{C \in Q^k} w_C x_C & & & & \\
\text{s.t.} & \sum_{C: v \in C} x_C & = 1 & \forall v \in V & \rightarrow \pi_v & \\
& \sum_{C \in Q^k} x_C & \leq \lfloor n/S \rfloor & & \rightarrow \sigma & (RMP) \\
& \sum_{C \subseteq T_i} x_C & \leq \lfloor |T_i|/S \rfloor & i = 1..t & \rightarrow \sigma_i & \\
& x_C & \geq 0 & & \rightarrow \alpha_C & \\
& C \in Q^k & , |C| \geq S & & &
\end{array}
$$

$$
\begin{array}{llllllll}
\max & \sum_{v=1}^{n} \pi_v & + \lfloor \frac{n}{S} \rfloor \sigma & + \sum_{i=1}^{t} \lfloor \frac{|T_i|}{S} \rfloor \sigma_i & & & & \\
\text{s.t.} & \sum_{v \in C} \pi_v & + \sigma & + \sum_{i: C \subseteq T_i} \sigma_i & + \alpha_C & = w_C & \forall C \in Q^k & (RMD) \\
& & \sigma & & & \leq 0 & & \\
& & & \sigma_i & & \leq 0 & i = 1..t & \\
& & & & \alpha_C & \geq 0 & \forall C \in Q^k &
\end{array}
$$

Computational Results

number of vertices max num of partitions	21-23 5	41-43 10	61-63 15	81-83 20	101-103 25
Heuristic Alg.					
Gap	5.30%	7.57%	7.46%	9.88%	8.21%
Time	0.0056	0.0114	0.0213	0.0337	0.0490
Root Node					
Total Instances	15	15	13	14	4
Solved exactly	11	14	8	5	0
Better Solution	11	14	13	14	4
Gap	0.27%	0.01%	0.98%	1.12%	1.22%
Time	2.41	16.47	188.48	826.04	1791.87
LPs solved	8	15	38	61	72
Total Cuts	2	6	19	35	42
Total Columns	107	200	320	440	528
Initial Columns	65	132	195	260	334
Columns by IP	1	2	8	12	18
Total IP Time	2.27	15.80	184.08	798.43	1718.75
Avg IP Time	1.09	5.14	20.28	62.15	92.91
B&B run					
total instances	4	1	5	9	4
Solved exactly	4	1	4	3	0
Better Solution	1	0	3	7	1
Gap	0%	0%	0.18%	0.36%	0.58%
Time	2.90	17.15	273.50	1529.98	4012.68
Nodes	2	1	3	6	10
Final columns	108	200	327	462	560

Table 1: Results on Random Geometric Problems for $S = 4$