MATP6640/DSES6770 Linear Programming, Homework 4.

Due: Thursday, March 20, 2008.
10% penalty for each day late.

A two-stage stochastic programming problem with recourse can be stated as minimizing a function $f(x) = E_\xi(g(x,\xi))$ over $x \in X$ for some set X, where ξ is a random variable with a known probability distribution. Assume ξ has a large, possibly infinite, set of possible realizations. Let $v^* := \min_{x\in X} f(x)$.

1. Let $\xi_1, \ldots, \xi_k$ be a random sample of realizations of the random variable ξ. Let

$$\hat{v}_k^* := \min_{x\in X}\{\hat{f}_k(x) := \frac{1}{k}\sum_{i=1}^{k} g(x,\xi_i)\}.$$

   This corresponds to assuming that the k scenarios are equally likely, and that they give the whole set of possible scenarios. Let $\hat{x}_k^*$ be an optimal solution to this problem. Show that the expected value of $\hat{v}_k^*$ is no larger than v^*.

2. Let $\xi_1, \ldots, \xi_l$ be a random sample of realizations of the random variable ξ. Let

$$\bar{f}_l(x) := \frac{1}{l}\sum_{i=1}^{l} g(x,\xi_i).$$

   Show that the expected value of $\bar{f}_l(x)$ is at least v^* for any $x \in X$.

3. It is desired to split some money between four different asset classes: domestic stocks, international stocks, bonds, and money market funds. To create some diversification, at least 10% and no more than 35% of the money is invested in each asset class. It is desired to maximize the expected value of the portfolio after one year. Various scenarios S are possible over the year, with scenario $s \in S$ having probability p_s. Let x_i denote the proportion of assets invested in class i, $i = 1, \ldots, 4$. Let y_s denote the proportional increase in the value of the portfolio at the end of the year under scenario s. Let r_i^s equal the value of one dollar invested in asset class i at the end of the year if scenario s is realized. The problem of maximizing the expected wealth at the end of the year can be expressed as an equivalent minimization problem as follows:

$$\begin{array}{llcrcll} \min_{x,y} & & & -\sum_{s\in S} p_s y_s & & & \\ \text{subject to} & \sum_{i=1}^4 x_i & & & = & 1 & \\ & \sum_{i=1}^4 r_i^s x_i & - & y_s & = & 1 & \forall s \in S \\ & & 0.1 \le & x_i & \le & 0.35 & i = 1,\ldots,4 \end{array} \qquad (SP)$$

   (In the notation of the introduction, we have $f(x) = -\sum_{s\in S} p_s(\sum_{i=1}^4 r_i^s x_i - 1)$.)

   Assume we have sampled the following scenarios, which are equally likely:

s	r_1^s	r_2^s	r_3^s	r_4^s
1	1.10	1.22	1.07	1.03
2	1.30	1.20	1.02	1.03
3	0.75	0.80	1.11	1.05

Use the approximation of question 1 to derive an expected lower bound on the optimal value of (SP), using AMPL or otherwise. Show that the optimal solution is $\hat{x}^* = (0.2, 0.35, 0.35, 0.1)$.

4. Returning to the setting of question 3, assume we have sampled the following equally-likely scenarios:

s	r_1^s	r_2^s	r_3^s	r_4^s
1	1.10	1.22	1.07	1.03
2	1.30	1.20	1.02	1.03
3	0.75	0.80	1.11	1.05
4	1.08	1.08	1.05	1.03
5	0.95	0.90	1.10	1.04
6	1.24	1.33	0.98	1.01
7	1.21	1.25	1.10	1.05
8	0.80	0.70	1.09	1.05
9	1.10	1.15	1.06	1.02
10	1.24	1.21	1.08	1.04

Take the optimal $\hat{x}$ from question 3. Calculate the corresponding expected upper bound $\bar{f}(x)$ from question 2. What is the solution to the problem in question 1 for this set of scenarios?

5. Consider a multicommodity network flow problem with three commodities on a graph $G = (V, E)$ with five nodes and seven edges. The seven edges and their costs and capacities are as follows:

edge	$(1,2)$	$(1,5)$	$(2,3)$	$(2,4)$	$(2,5)$	$(3,4)$	$(4,5)$
cost	8	11	7	10	6	6	8
capacity	25	40	35	50	40	25	50

The costs are identical for each commodity. 25 units of commodity A need to be moved from node 1 to node 3. 20 units of commodity B need to be moved from node 5 to node 3. 20 units of commodity C need to be moved from node 4 to node 1. The current set of paths consists of $1 \to 2 \to 3$ and $1 \to 5 \to 4 \to 3$ for commodity A, $5 \to 2 \to 3$ for commodity B, and $4 \to 5 \to 1$ for commodity C. Use the path based approach described in class to show that this is not an optimal set of paths. What is the optimal solution?

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