Exam Review

Chapter 11.9 - 11.11

1. Find the Maclaurin series for the following using (i) direct method (ii) know series.

(a) $f(x) = \frac{x^2}{1+x}$ (b) $f(x) = \sin(x^4)$

2. Find the first four non-zero terms for

$g(x) = \frac{e^x}{1-x}$

3. Find the indefinite integral $\int \frac{e^x}{x}$

4. Find $\int_0^{0.5} x^2 e^{-x^2} dx$ with an error of 0.001

Usefule list of identities

(1) $x^a x^n = x^{a+n}$

(2) $x^{-a} = \frac{1}{x^a}$

(3) $0! = 1$

(4) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $R=1$, $I=(-1,1)$

(5) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $R=\infty$, $I=(-\infty,\infty)$

(6) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $R=\infty$, $I=(-\infty,\infty)$

(7) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $R=\infty$, $I=(-\infty,\infty)$

(8) $\arctan x = \sum_{n=0}^{\infty} (-1) \frac{x^{2n+1}}{2n+1}$ $R=1$, $I=[1,1]$

Solutions
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(1) $f(x) = \frac{x^2}{1+x}$

(i)

let $g(x) = \frac{1}{1+x}$ $\qquad g(0) = 1 = (-1)^0 \cdot 0!$

$g'(x) = -\frac{1}{(1+x)^2}$ $\qquad g'(0) = -1 = (-1)^1 \cdot 1!$

$g''(x) = \frac{2}{(1+x)^3}$ $\qquad g''(0) = 2 = (-1)^2 \cdot 2!$

$g'''(x) = \frac{-6}{(1+x)^4}$ $\qquad g'''(0) = -6 = (-1)^3 \cdot 3!$

$\Rightarrow g^n(0) = (-1)^n (n!)$

$$\therefore \frac{1}{1+x} = g(0) + g'(0)x + g\frac{''(0)x^2}{2!} + \frac{g''(0)x^3}{3!} + \dots$$

$$= 1 - x + \frac{\cancel{2}x^2}{\cancel{2}} - \frac{\cancel{6}x^2}{\cancel{6}} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

since $f(x) = x^2 \cdot g(x) = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \underline{\sum_{n=0}^{\infty} (-1)^n x^{n+2}}$

(ii)

notice that $\frac{1}{1+x} = \frac{1}{1-(-x)}$

Using the power series representation of $\frac{1}{1-(-x)}$, we get $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

$$\Rightarrow \frac{x^2}{1+x} = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \underline{\sum_{n=0}^{\infty} (-1)^n x^{n+2}}$$

(b) $f(x) = \sin(x^4)$

(i) It would be really long to derive a direct Maclaurin series for $f(x) = \sin(x^4)$

let $f(y) = \sin y$. $\quad f(0) = \sin(0) = 0$

$f'(y) = \cos y \qquad f'(0) = \cos(0) = 1 = (-1)^0$

$f''(y) = -\sin y \qquad f''(0) = -\sin(0) = 0$ –

$f'''(y) = -\cos y \qquad f'''(0) = -\cos(0) = -1 = (-1)^1$

$f''''(y) = \sin y \qquad f''''(0) = \sin 0 = 0$

$\vdots \qquad \vdots$

notice only the odd derivatives have non-zero results.

$\Rightarrow f^n(0) = (-1)^n$ and only $2n+1$ terms survive

$\Rightarrow f(y) = \cancel{\frac{0 \cdot 1}{0!}} + \frac{1 \cdot y}{1!} \cancel{+ \frac{0 \cdot y^2}{2!}} - \frac{1 \cdot y^3}{3!} \cancel{+ \frac{0 \cdot y^4}{4!}} + \frac{1 \cdot y^5}{5!} - \ldots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$

since $f(y) = \sin(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$

$\Rightarrow \sin(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}$

(ii) Using the known formula for $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n+1}}{(2n+1)!}$

Substitute x^4 for every x and solve as in part (i)

(2) $g(x) = \frac{e^x}{1-x}$

recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{and} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) \qquad = \left(1 + x + x^2 + x^3 + \ldots\right)$$

$$\Rightarrow \frac{e^x}{1-x} = \left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \ldots\right)\left(1 + x + x^2 + x^3 + \ldots\right)$$

$$\begin{aligned} = \; & 1 + x + x^2 + x^3 + \ldots \\ & + x + x^2 + x^3 + x^4 + \ldots \\ & + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{2}x^5 + \ldots \\ & + \tfrac{1}{6}x^3 + \tfrac{1}{6}x^4 + \tfrac{1}{6}x^5 + \ldots \end{aligned}$$

$$= 1 + 2x + \tfrac{5}{2}x^2 + \tfrac{16}{6}x^3 + \ldots$$

(3) $\int e^x/x \, dx$

recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\Rightarrow \frac{e^x}{x} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{x}\left[1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \ldots\right]$$

$$= \frac{1}{x} + 1 + \tfrac{1}{2}x + \tfrac{1}{6}x^3 + \ldots$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Why did I do this? To avoid singularity at $n=0$ so that I can integrate term by term.

$$\Rightarrow \int \frac{e^x}{x} = \int \left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\right] dx = \underline{\left[\ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C\right]}$$

(4) recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(x^{-2})^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\Rightarrow x^2 e^{-\cancel{x^{-2}}} = x^2 \sum_{n=0}^{\infty} \frac{(-1) x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

$$\Rightarrow \int_0^{0.5} \cancel{x}^2 e^{-x^2} dx = \int_0^{0.5} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!} dx$$

We can integrate term by term by continuity

$$= \sum_{n=0}^{\infty} \int_0^{1/2} (-1)^n \frac{x^{2n+2}}{n!} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n+3}}{(2n+3)n!} \right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+3}}{(2n+3)n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+3)n!\, 2^{2n+3}}$$

$$= (-1)^0 \frac{1}{3} \cdot \frac{1}{2^3} - \frac{1}{5} \frac{1}{2^5} + \frac{1}{7 \cdot 2 \cdot 2^7} - \ldots$$

$$= \frac{1}{24} - \frac{1}{160} + \frac{1}{1792}$$

since we are looking for an error of $\frac{1}{1000}$, we can stop at $n = 2$ since $\frac{1}{1792} < \frac{1}{1000}$.

$$\therefore \int_0^{0.5} x^2 e^{-x^2} dx = \frac{1}{24} - \frac{1}{160} \approx \underline{0.0354}$$