Phase transitions to Super-rotation in a Coupled Fluid - Rotating Sphere System

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Summary. A family of spin-lattice models are derived as convergent finite dimensional approximations to the rest frame kinetic energy of a barotropic fluid coupled to a massive rotating sphere. The angular momentum of the fluid component changes under complex torques that are not resolved and the kinetic energy of the fluid is not a conserved Hamiltonian in these models. These models are used in a statistical equilibrium formulation for the energy - relative enstrophy theory of the coupled barotropic fluid - rotating sphere system, known as Kac's spherical model, to study the interactions between the energy cascade to large scales and angular momentum transfer. Exact solution of this model provides critical temperatures and amplitudes of the ground modes - super-rotating solid body flows - in the BECondensed phase.

Key words: Coupled barotropic flows, super-rotation, phase transition, exactly-solvable spherical model, non-Hamiltonian setting, angular momentum

1 Introduction

This paper offers a short review [16] of some recent advances as well as specific new results [5], [7], [6], [17], [11], [10] in the application of equilibrium statistical mechanics to complex geophysical and astrophysical flows [1], [15] by the author and his collaborators. These applications - based squarely on the physical [9] and mathematical [16] significance of the robust quantities energy, enstrophy and angular momentum - include but are not restricted to the super-rotation of the Venusian middle atmosphere [3]. An important overall aim of our results is the extension of statistical equilibrium theories to macroscopic flows that lack some of the basic properties previously assumed to be necessary such as a conserved Hamiltonian and selected invariants - global angular momentum and higher order vorticity moments. Specific problems to which we address this review include global scale flows coupled to a rotating solid sphere by complex torque mechanisms.

Decades of research show that the standard statistical equilibrium approach have produced some qualified scientific progress in the simplest cases

such as flows in a periodic square, in the unbounded plane and uncoupled flows on a sphere [13], [4], [12], [1]. We will discuss some of the reasons for current dis-satisfaction with this standard approach [23].

Fjortoft's and later Kraichnan's identification of energy inverse cascades in nearly inviscid quasi-2D turbulence - a statistically stationary result - renewed interest in Onsager's approach [13] of using a canonical Gibbs ensemble to model energy transfer in unbounded and periodic domains [24], [4], [1]. A microcanonical ensemble has sometimes been used [2] to avoid questions about the physicality of the energy reservoir in the canonical approach for unbounded and periodic domains. There is no question that a canonical approach should be used for bounded flows which are not mechanically nor energetically isolated.

However, the Hamiltonian vortex gas models - with fixed numbers of particles - conserve circulation and angular momentum and for this reason, they are unsuitable for modelling quasi-2D turbulent flows with complex boundaries such as no-slip conditions and in particular flows coupled to a sphere. It is easy to see that in this class of quasi-2D flows, angular momentum and all the vorticity moments - except circulation - are not conserved in the fluid component of the coupled system.

It is therefore not surprising that a statistical equilibrium model based on the mean field theory of point vortices failed to predict the number and types of coherent vortices in recent significant experiments of decaying quasi-2D turbulent flows in no-slip rectangular domains reported by Maassen, Clerx and van Heijst [23]. This series of experiments beginning with Sommeria [22] have produced remarkable observations of the role of angular momentum in quasi-2D turbulence in complex bounded domains.

We turn now to the matter of enstrophy constraints. Because of the doubly canonical form of its Gibbs ensemble, the classical energy-enstrophy theories are equivalent to the Gaussian model and therefore exactly-solvable. This fact which is one of the reasons for its wide acceptance, is also the source of a serious disadvantage - it is not well defined at low temperatures [5]. Other theories - based on the explicit conservation of all vorticity moments [21], [19] - have been proposed recently and have spawned a long series of papers in the open literature. Again, the same reasoning as above shows that these models - due to a microcanonical approach in one case and to the Hamiltonian conservation of all vorticity moments in both - cannot be used to predict features in quasi-2D turbulent flows coupled to a sphere.

Previous work [1] based on applying a statistical equilibrium approach to the Barotropic Vorticity Model (BVM) - for which angular momentum is conserved - have naturally failed to detect any phase transitions to super- and sub-rotating flow states because they are based on physical models (BVM) which do not allow for the exchange of angular momentum between the atmosphere and the solid planet - a key mechanism in any model that hopes to capture aspects of the enigmatic super-rotation problem in planetary atmospheres.

What we offer here differs from previous work - three key ideas characterize the projects in the author's research programme: (A) formulation of correct and solvable statistical mechanics theories of geophysical flows based on energy, enstrophy, total circulation and non-conservation of angular momentum, (B) effective computer simulations of these vortical systems with multiple time and length scales on a small, economical Linux cluster, and (C) exact closed form solutions of these models [17].

The specific implementation of microcanonical enstrophy constraints in this approach leads to (C) - exact solutions of the resulting theories using the Kac-Berlin method [14] of steepest descent for spherical models. The main point discussed below in further detail is that these spherical models fix the low temperature problems of the classical energy-enstrophy theories [4] and yet are solvable in closed form. Crucial to the discussion here however is item (A) - by correct statistical mechanics theories for the class of quasi-2D problems in complex domains, we mean precisely that unlike all previous work, this approach is based on energy functionals which are neither the Hamiltonian nor Lagrangian for the flow [6]. Indeed, it is easy to surmise that if there are local in time equations of motion for the flow, these PDEs are not Hamiltonian in form because energy and angular momentum are not conserved in the fluid component of the coupled fluid-sphere systems. This situation does not present serious problems for the statistical mechanics approach in general - it requires only a partition function based on a energy functional (not necessarily Hamiltonians) and constraints that are defined in overall phase space. Topological arguments - Stokes' theorem - imply the conservation of circulation. Arguments based on the vanishing of enstrophy dissipation in the zero viscosity limit- known to be a rigorous result in quasi-2D stationary flows with finite enstrophy - and the fact that the enstrophy is the (square of the) square-norm of the vorticity in the natural Hilbert space for these problems, will be given for fixing the enstrophy. Jung, Morrison and Swinney [9] have argued experimentally that energy and enstrophy are the most important quantities in the application of statistical equilibrium theories to macroscopic flows. Thus, the only relevant quantities in this approach are energy of flow, angular momentum, enstrophy and circulation - a conclusion that is based on the above analysis of the physics of quasi-2D stationary flows in complex domains.

We will derive in this paper such a generalized energy functional for a coupled geophysical flow that is not a Hamiltonian nor a Lagrangian [20].

2 Coupled Barotropic Fluid - Rotating solid Sphere Model.

Consider the system consisting of a rotating massive rigid sphere of radius R , enveloped by a thin shell of non-divergent barotropic fluid. The barotropic flow is assumed to be inviscid, apart from an ability to exchange angular

momentum and kinetic energy with the infinitely massive solid sphere through unresolved torque and Ekman pumping mechanisms. We also assume that the fluid is in radiation balance and there is no net energy gain or loss from insolation. This provides a crude model of the complex planet - atmosphere interactions, including the enigmatic torque mechanism responsible for the phenomenon of atmospheric super-rotation - one of the main applications motivating this work.

For a geophysical flow problem concerning super-rotation on a spherical surface there is little doubt that one of the key parameters is angular momentum of the fluid. In principle, the total angular momentum of the fluid and solid sphere is a conserved quantity but by taking the sphere to have infinite mass, the active part of the model is just the fluid which relaxes by exchanging angular momentum with an infinite reservoir. It is also clear that a 2d geophysical relaxation problem such as this one will involve energy and enstrophy. The rest frame energy of the fluid and sphere is conserved. Since we have assumed the mass of the solid sphere to be infinite, we need only keep track of the kinetic energy of the barotropic fluid - in the non-divergent case, there is no gravitational potential energy in the fluid because it has uniform thickness and density, and its upper surface is a rigid lid.

In a nutshell, we need to find a suitable set of constraints for the obvious choice of objective functional, namely rest frame kinetic energy of flow in the coupled model. The choice of auxillary conditions or constraints is not apriori obvious.

We will use spherical coordinates - $\cos\theta$ where θ is the colatitude and longitude ϕ . The total vorticity is given by

$$q(t; \cos\theta, \phi) = \Delta\psi + 2\Omega \cos\theta \quad (1)$$

where $2\Omega \cos\theta$ is the planetary vorticity due to spin rate Ω and $w = \Delta\psi$ is the relative vorticity given in terms of a relative velocity stream function ψ and Δ is the negative of the Laplace-Beltrami operator on the unit sphere S^2 . Thus, a relative vorticity field, by Stokes' theorem, has the following expansion in terms of spherical harmonics,

$$w(x) = \sum_{l \geq 1, m} \alpha_{lm} \psi_{lm}(x). \quad (2)$$

A key property that will be established later is that the mode $\alpha_{10}\psi_{10}(x)$ contains all the angular momentum in the relative flow with respect to the frame rotating at the fixed angular velocity Ω of the sphere.

2.1 Physical quantities of the coupled barotropic vorticity model

The rest frame kinetic energy of the fluid expressed in a frame that is rotating at the angular velocity of the solid sphere is

$$\begin{aligned}
H_T[q] &= \frac{1}{2} \int_{S^2} dx [(u_r + u_p)^2 + v_r^2] \\
&= -\frac{1}{2} \int_{S^2} dx \psi q + \frac{1}{2} \int_{S^2} dx u_p^2
\end{aligned}$$

where u_r, v_r are the zonal and meridional components of the relative velocity, u_p is the zonal component of the planetary velocity (the meridional component being zero since planetary vorticity is zonal), and ψ is the stream function for the relative velocity. Since the second term $\frac{1}{2} \int_{S^2} dx u_p^2$ is fixed for a given spin rate Ω , it is convenient to work with the pseudo-energy as the energy functional for the model,

$$\begin{aligned}
H[w] &= -\frac{1}{2} \int_{S^2} dx \psi q = -\frac{1}{2} \int dx \psi(x) [w(x) + 2\Omega \cos \theta] \\
&= -\frac{1}{2} \int dx \psi(x) w(x) - \Omega \int dx \psi(x) \cos \theta
\end{aligned}$$

Relative vorticity circulation in the model is fixed to be $\int w dx = 0$, which is a direct consequence of Stokes' theorem on a sphere. It is easy to see that the kinetic energy functional H is not well-defined without the further requirement of a constraint on the size of its argument, the relative vorticity field $w(x)$. A natural constraint for this quantity is therefore its square norm or relative enstrophy so as to carry out the analysis in the Hilbert space, $L^2(S^2)$. Thus, we see that our choice of the relative enstrophy microcanonical constraint, is natural from the vanishing dissipation standpoint and also required for a rigorous variational analysis.

The second term in the energy is equal to 4Ω times the variable angular momentum density of the relative fluid motion and has units of m^4/s . The physical angular momentum, given by

$$\rho \int_{S^2} dx w \cos \theta = \rho \langle w, \cos \theta \rangle, \quad (3)$$

implies that the only mode in the eigenfunction expansion of w that contributes to its net angular momentum is $\alpha_{10}\psi_{10}$ where $\psi_{10} = a \cos \theta$ is the first nontrivial spherical harmonic; it has the form of solid-body rotation vorticity.

All other moments of the vorticity $\int dx q^n$ are not considered here.

3 Spin-Lattice Approximation

Given the well known fact that Gibbs' canonical ensemble and the corresponding partition function for the spherical model - to be discussed in detail below - are closely related to path-integrals and therefore extremely complex mathematical objects, a rational approximation procedure based on finite dimensional spin-lattice models or something similar, will have to be devised

to simulate their critical phenomenology on the computer as well as to solve them by analytical means. Such a rational approximation scheme must satisfy two basic requirements when the size or order of the approximation is taken to infinity: (A) the resulting family of finite dimensional models converge to the correct energy functional and constraints of the problem and (B) the thermodynamic limit - in this case, the nonextensive continuum limit - of this family of approximate models exists. (A) is shown in [16] to be true for the family of spin-lattice models given next. (B) turns out to be true because exact solutions to the spherical models - obtained by the Kac-Berlin method of steepest descent in [17] and summarized below - yield valid free energy expressions in terms of the associated saddle points in the nonextensive continuum limit.

Under the piecewise constant approximation for the relative vorticity w , based on Voronoi cells on a lattice, the truncated energy takes the standard form of a spin lattice model,

$$H_N = -\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N J_{jk} s_j s_k - \sum_{j=1}^N F_j s_j. \quad (4)$$

with the interactions

$$J_{jk} = \frac{16\pi^2}{N^2} \ln |1 - x_j \cdot x_k|$$

and the external fields

$$F_j = -\frac{2\pi}{N} \Omega ||\cos \theta||_2 \psi_{10}(x_j)$$

where $||\cos \theta||_2$ is the L_2 norm of the function $\cos \theta$, and the spherical harmonic ψ_{10} which represents the relative vorticity of solid-body rotation. The truncated relative enstrophy given by

$$\Gamma_N = \frac{4\pi}{N} \sum_{j=1}^N s_j^2 \quad (5)$$

is fixed and the truncated circulation given by

$$TC_N = \frac{4\pi}{N} \sum_{j=1}^N s_j$$

is fixed at zero.

4 Solution of the Spherical Model.

We now summarize the exact solution of the spherical model [14] for barotropic flows in the inertial frame and refer the reader to the literature for details [17]. The partition function for the spherical model has the form

$$Z_N \propto \int D(\mathbf{s}) \exp(-\beta H_N(\mathbf{s})) \delta \left(\Omega \frac{N}{4\pi} - \sum_{j=1}^N \mathbf{s}_j \cdot \mathbf{s}_j \right) \quad (6)$$

where the integral is a path-integral taken over all the microstates $\mathbf{s}$ with zero circulation. In the thermodynamic or continuum limit considering the integral as $N \rightarrow \infty$, this partition function can be calculated using Laplace's integral form,

$$\begin{aligned} Z_N &\propto \int D(\mathbf{s}) \exp(-\beta H_N(\mathbf{s})) \delta \left(\Omega \frac{N}{4\pi} - \sum_{j=1}^N \mathbf{s}_j \cdot \mathbf{s}_j \right) \\ &= \int D(\mathbf{s}) \exp(-\beta H_N(\mathbf{s})) \left(\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp \left(\eta \left(\Omega \frac{N}{4\pi} - \sum_{j=1}^N \mathbf{s}_j \cdot \mathbf{s}_j \right) \right) \right) \end{aligned} \quad (7)$$

Solving these Gaussian integrals requires first writing the site vorticities $\mathbf{s}_j$ in terms of the spherical harmonics $\{\psi_{l,m}\}_{l=1}^{\infty}$, which are the natural Fourier modes for Laplacian eigenvalue problems on S^2 with zero circulation, and using this to diagonalize the interaction in H_N ,

$$\begin{aligned} Z_N &\propto \int D(\alpha) \exp \left(-\frac{\beta}{2} \sum_{l=1}^{\infty} \sum_{m=-l}^l \lambda_{lm} \alpha_{lm}^2 \right) \\ &\times \left(\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp \left(\eta N \left(1 - \frac{4\pi}{\Omega} \sum_{l=1}^{\infty} \sum_{m=-l}^l \alpha_{lm}^2 \right) \right) \right) \end{aligned} \quad (9)$$

The eigenvalues of the Green's function for the Laplace-Beltrami operator on S^2 are

$$\lambda_{l,m} = \frac{1}{l(l+1)}, \quad l = 1, 2, \dots, \sqrt{N}, m = -l, -l+1, \dots, 0, \dots, l \quad (10)$$

and $\alpha_{l,m}$ are the corresponding amplitudes so that

$$s(\mathbf{x}_j) = \sum_{l=0}^{\sqrt{N}} \sum_{m=-l}^l \alpha_{l,m} \psi_{l,m}(\mathbf{x}_j) \quad (11)$$

for each of the mesh sites x_j .

Next we exchange the order of integration in equation 9. This is allowed provided that a is positive and is chosen large enough so that the integrand is absolutely convergent. Rescaling the inverse temperature $\beta' N = \beta$ yields

$$Z_N \propto \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp \left(\eta N \left(1 - \frac{4\pi}{\Omega} \sum_{l=1} \sum_{m=-l}^l \alpha_{l,m}^2 \right) - \frac{\beta' N}{2} \sum_{m=-1}^1 \lambda_{1,m} \alpha_{1,m}^2 \right) \\ \times \int_{l \geq 2} D(\alpha) \exp \left(- \sum_{l=2} \sum_{m=-l}^l \left(\frac{\beta' N \lambda_{l,m}}{2} + N \eta \frac{4\pi}{\Omega} \right) \alpha_{l,m}^2 \right). \quad (12)$$

We can explicitly solve this inner integral because it is the product of a collection of Gaussian integrals. So

$$\int_{l \geq 2} D(\alpha) \exp \left(- \sum_{l=2} \sum_{m=-l}^l \left(\frac{\beta' N \lambda_{l,m}}{2} + N \eta \frac{4\pi}{\Omega} \right) \alpha_{l,m}^2 \right) \\ = \prod_{l=2}^{\sqrt{N}} \prod_{m=-l}^l \left(\frac{\pi}{N \eta \frac{4\pi}{\Omega} + \frac{\beta' N}{2} \lambda_{l,m}} \right)^{1/2} \quad (13)$$

provided this physically important conditions holds:

$$\frac{\beta' \lambda_{l,m}}{2} + \eta \frac{4\pi}{\Omega} > 0, \quad l = 2, 3, \dots, \sqrt{N}, \quad m = -l, -l+1, \dots, 0, \dots, l \quad (14)$$

So we can now simplify equation 12 as

$$Z_N \propto \int_{a-i\infty}^{a+i\infty} d\eta \exp \left(N \left[\begin{array}{c} \eta \left(1 - \frac{4\pi}{\Omega} \sum_{m=-1}^1 \alpha_{1,m}^2 \right) \\ - \frac{\beta'}{2} \sum_{m=-1}^1 \lambda_{1,m} \alpha_{1,m}^2 \\ - \frac{1}{2N} \sum_{l=2} \sum_m \log \left(N \eta \frac{4\pi}{\Omega} + \frac{\beta' N}{2} \lambda_{l,m} \right) \end{array} \right] \right) \quad (15)$$

which we can cast in a form suitable for the saddle point method or method of steepest descent,

$$Z \propto \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp (N F(\eta, \Omega, \beta)). \quad (16)$$

In the thermodynamic limit as $N \rightarrow \infty$, where the free energy per site – after separating out the 3-fold degenerate ground states $\psi_{1,0}, \psi_{l,1}, \psi_{l,-1}$ – is, modulo a factor of $-\beta'$, given by

$$F(\eta, \Omega, \beta') = \eta \left(1 - \frac{4\pi}{\Omega} \sum_{m=-1}^1 \alpha_{1,m}^2 \right) - \frac{\beta'}{2} \sum_{m=-1}^1 \lambda_{1,m} \alpha_{1,m}^2 \\ - \frac{1}{2N} \sum_{l=2} \sum_m \log \left(N \eta \frac{4\pi}{\Omega} + \frac{\beta' N}{2} \lambda_{l,m} \right) \quad (17)$$

The saddle point condition is

$$0 = \frac{\partial F}{\partial \eta} = \left(1 - \frac{4\pi}{\Omega} \sum_{m=-1}^1 \alpha_{1,m}^2 \right) - \frac{2\pi}{\Omega} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^l \left(N \eta \frac{4\pi}{\Omega} + \frac{\beta' N}{2} \lambda_{l,m} \right)^{-1}. \quad (18)$$

To close the system we need a set of three additional constraints; these are given by the equations of state for $m = -1, 0$, and 1 :

$$0 = \frac{\partial F}{\partial \alpha_{1,m}} = \left(\frac{8\pi\eta}{\Omega} + \beta' \lambda_{1,m} \right) \alpha_{1,m} \quad (19)$$

The last three equations have as solutions

$$\alpha_{1,m} = 0 \quad \text{or} \quad \frac{8\pi\eta}{\Omega} + \beta' \lambda_{1,m} = 0, \text{ for each } m. \quad (20)$$

This means that in order to have nonzero amplitudes in at least one of the ground or condensed states, which are the only ones to have angular momentum,

$$\frac{4\pi\eta}{\Omega} = -\frac{\beta'}{4} \quad (21)$$

which implies that the inverse temperature must be negative,

$$\beta' < 0. \quad (22)$$

The Gaussian condition, equation 14, on the modes with $l = 2$

$$\frac{\beta'}{12} - \frac{\beta'}{4} > 0 \quad (23)$$

can only be satisfied by a negative temperature, $\beta' < 0$, when there is any energy in the angular momentum containing ground modes.

When we substitute this nonzero solution into the saddle point equation it yields

$$0 = \left(1 - \frac{4\pi}{\Omega} \sum_{m=-1}^1 \alpha_{1,m}^2 \right) - \frac{4\pi}{\Omega} \frac{T}{N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^l \left(\lambda_{l,m} - \frac{1}{2} \right)^{-1} \quad (24)$$

$$= \left(1 - \frac{4\pi}{\Omega} \sum_{m=-1}^1 \alpha_{1,m}^2 \right) - \frac{T}{T_c} \quad (25)$$

where the critical inverse temperature is negative, has a finite large N limit, and is inversely proportional to the relative enstrophy Ω ,

$$-\infty < \beta'_c = \frac{4\pi}{\Omega N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^l \left(\lambda_{l,m} - \frac{1}{2} \right)^{-1} < 0 \quad (26)$$

The saddle point equation gives us a way to compute the equilibrium amplitudes of the ground modes for temperatures hotter than the negative critical temperature T_c . For temperatures T so that $T_c < T < 0$,

$$\sum_{m=-1}^1 \alpha_{1,m}^2(T) = \frac{\Omega}{4\pi} \left(1 - \frac{T}{T_c}\right) \quad (27)$$

This argument shows that at positive temperatures, there cannot be any energy in the solid-body rotating modes. In other words, there is no phase transition at positive temperatures. This is the spin-lattice representation of the self-organization of barotropic energy into a large-scale coherent flow at very high energies in the form of symmetry-breaking Goldstone modes. These extremely high energy ground modes carry a nonzero angular momentum that can be directed along an arbitrary axis - this problem is formulated in the inertial frame with planetary spin $\Omega = 0$. The new book by Lim and Nebus [16] provides many more details of the work reviewed and presented here.

Another project to simulate and analyze the statistical equilibria of a layer of divergent fluid coupled to a massive rotating sphere using the spin-lattice models that generalize those given here is under way and have produced convincing preliminary results that initially nondivergent flow states are thermodynamically unstable to divergent perturbations [8]. The key fact, however, from these recent numerical experiments [8] - overall divergent statistical equilibria have been found to have nondivergent parts (relative vorticity) that are close to super- and sub-rotating solid-body barotropic flows - appears to justify our analysis of the spherical model for the coupled barotropic fluid - sphere system.

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