

Finance Seminar 2 september 30 2007

Professor Chjan C. Lim

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Abstract

“Students who successfully complete the one year sequence of financial math and simulations courses with Chjan Lim are naturally exposed to basic and deep financial ideas and lines of reasoning, and will be equipped to deal with possibly intricate and delicate financial questions that arise as a matter of course in their future work in the finance industry.”

Seminar 2

Instructor: Professor Chjan C. Lim

Seminar 2 concerns the relationship - within the binary tree / Cox-Rubinstein model - between real-world probability $(q, 1 - q)$, the log-normal model of stock prices based on mean return μ and volatility σ through the SDE :

$$dS = \nu S dt + \sigma S dW,$$

and the risk-neutral probability $(p, 1 - p)$ and the Martingale-based option prices. Students were asked to provide solutions (u, d, q) in terms of $\mu, \delta t, \sigma$ to the pair of equations below.

The real world probability $(q, 1 - q)$ can be defined in terms of the single-period binary model, with random variable $X = u, d$:

$$\begin{aligned} qu + (1 - q)d &= E[X] = e^{\mu\delta t} \\ qu^2 + (1 - q)d^2 - (qu + (1 - q)d)^2 &= var(X) = \sigma^2\delta t. \end{aligned}$$

This appears to be an incomplete set of equations requiring an apriori assumption perhaps about the relationship between u and d .

Several partial solutions and one relatively complete one were presented - they are

(1) assume (Huang, Wang, Hua, Birthright, Lehman, Tono) $ud = 1$ and show $u = e^{\sigma\sqrt{\delta t}}$ with Huang's approach based on the full exponential expression,

(2) assume (Iqbal) $q = 1/2$ or equivalently $u + d = 2$ and $\mu = 0$ to get $u, d = 1 \pm \sigma\sqrt{\delta t}$,

(3) no symmetry assumption (Cheong) to get after power -series expansion and dropping terms of $O(\delta t^2)$, solutions u, d that depends on both μ and σ .

(4) symmetry properties $u + d = 2$ was justified (Sharma) by assuming that the stock volatility σ^2 must be the same in both the real-world and risk-neutral pictures.

First 3 sets of attempts give correct values of u and d because they are not unique but did not show an understanding of the deep financial reasons why a symmetry assumption such as $u + d = 2$ can be justified - indeed the most general set is in (3) with no symmetry assumptions and thus no assumption on μ and q ; solution (2) gives correct values of u, d when $\mu = 0$ and $q = 1/2$ without justifying these 2 assumptions - see below for explanation why these assumptions are financially and mathematically sound; and solutions (1) give correct values of u, d but the key assumption $ud = 1$ was not financially explained - it is not difficult to see how one can justify apriori (from Black-Scholes) $ud = 1$, which yields exponential form of $u, d = e^{\pm\sigma\sqrt{\delta t}}$ and gives

$$q = \frac{e^{\mu\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}}$$

that is meaningful provided $e^{\mu\delta t} < e^{\sigma\sqrt{\delta t}}$. The justification is along the same lines explained below with one important exception. between q and μ can now be used to change both since the real-world q does not enter the calculation

of option prices - it is therefore convenient to choose $\mu = 0$ and let $q = \frac{1 - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}}$.

The linear expansion form of $u, d = 1 \pm \sigma\sqrt{\delta t}$ gives $u + d = 2$ and in approximation, $u, d = e^{\pm\sigma\sqrt{\delta t}}$ which leads to the more elegant binary tree - and therefore also chosen by Cox-Rubinstein - with consequence that the risk-neutral probability $(p, 1 - p)$,

$$\begin{aligned} pu + (1 - p)d &= pe^{\sigma\sqrt{\delta t}} + (1 - p)e^{-\sigma\sqrt{\delta t}} = e^{r\delta t} \\ p &= (e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}) / (e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}) \end{aligned}$$

is meaningful if

$$e^{r\delta t} < e^{\sigma\sqrt{\delta t}}.$$

The most complete and original solution - one that showed a real understanding of the Martingale /SDE properties - starts with the assumption that the volatility must be the same in both pictures, resulting in 4 equations for 4 unknowns u, d, q, p in terms of $\mu, \sigma, \delta t$ and r . This solution requires *interest* rate r but is otherwise equivalent to the assumptions that the deterministic drift in stock price μ and the real-world probability q are factored into the Black-Scholes -Merton- Cox-Rubinstein option pricing theory in a way that does not change the evaluation of claims. Sharma's solution is given below.

Since the Cox-Rubinstein Binary Tree model behind this seminar is based on the log-normal or geometric Brownian motion model of stock prices - more on these topics later in the course - and the Black-Scholes-Merton option pricing theory, it naturally depends, as Martingales do, on risk-neutral probability $(p, 1 - p)$:

$$E[S_1] = e^{r\delta t} S_0$$

or

$$E[\bar{S}_1] = e^{-r\delta t} E[S_1] = \bar{S}_0$$

equivalently,

$$pu + (1 - p)d = e^{r\delta t}$$

which gives option prices that depends on $r, u, d, \delta t, K, S_0$ but independent of μ and q . This is the key assumption I wanted you all to arrive at, or since I emphasized it in class, to use as the starting point of the seminar.

Lets us examine the delicate reasoning that follows in the complete resolution of the seminar problem - (i) since Cox-Rubinstein option prices does

not depend on μ , we set $\mu = 0$ to get u, d which then gives us p . Setting $\mu = 0$ gives

$$qu + (1 - q)d = 1$$

and (ii) any solution $(u, d; q)$ that satisfies this and the variance equation, will generate correct option prices - so set $q = 1/2$ since option prices are independent of real-world probability to get one particularly nice solution,

$$u + d = 2; \quad q = 1/2.$$

This yields the answer $u = 1 + \sigma\sqrt{\delta t} \simeq e^{\sigma\sqrt{\delta t}}$.

We will sketch a proof that option prices are independent of μ and q . It is easily shown - and will be done more carefully in lecture - that in the limit $\delta t \rightarrow 0$, the real-world lognormal model for stock prices is given by

$$S_t = S_0 \exp(\mu t + \sigma\sqrt{t}N(0, 1))$$

under true real-world probability of uptick $q = 1/2$ with true

$$u, d = \exp(\mu t \pm \sigma\sqrt{t});$$

a different choice of q' with the same u, d will change the deterministic drift to $\mu' = f(q')$ away from the true mean return μ in the realworld lognormal model as $\delta t \rightarrow 0$.

On the other hand, under the risk-neutral probability

$$\begin{aligned} p &= \frac{1}{2} [1 - \sqrt{\delta t}A] \\ A &= \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma} \end{aligned}$$

for uptick u , it follows after a lengthy straightforward calculation that the lognormal model for stock price in the limit $\delta t \rightarrow 0$, is given by

$$S_t = S_0 \exp\left(\sigma\sqrt{t}N(0, 1) + \left(r - \frac{1}{2}\sigma^2\right)t\right)$$

with a different deterministic drift $r - \frac{1}{2}\sigma^2$ but the same volatility σ . The last step is the heuristic calculation of option (call) price in continuous time,

$$\begin{aligned} C_0 &= e^{-rT} E_P[(S_T - K)_+] \\ &= E_P[(e^{-rT} S_T - e^{-rT} K)_+] \\ &= E_P[(S_0 \exp(\sigma\sqrt{T}N(0, 1) + (-\frac{1}{2}\sigma^2)T) - e^{-rT} K)_+] \end{aligned}$$

where K is the strike price and T is the time to expiry of the European call on the stock S_t , which is clearly independent of q .

This shows that we can either keep true u, d, μ, σ, q in discrete time and have a continuous time drift $\mu' = f(q)$ which happens to be $\mu' = \mu$ when $q = 1/2$ or keep true up(down) ticks u, d, μ, σ but alter q to q' and at the same time change the continuous time drift to $\mu' = f(q')$ in the altered real-world lognormal model, and still end up with the same C_0 . Since $\mu' = f(q')$ this implies that option prices are independent of the effective continuous time drift μ' . It follows that we can choose $q = 1/2$ to keep $\mu' = \mu$ as in Etheridge even if the true uptick probability is not $1/2$.

Since we started with a discrete time framework, it will be nice to show that option prices are also independent of the discrete time or binary model drift μ . This can be done despite the fact that the discrete time risk-neutral probability p depends explicitly on μ . In passing to the limit $\delta t \rightarrow 0$, C_0 is independent of μ as in Etheridge for example. If we had only the continuous time model, then there is nothing left to justify since option prices are independent of the continuous time real-world drift μ' .

Sharma's solution here: We make the following 2 assumptions:-

1. δt is a very small quantity. So $e^{x\delta t}$ can be approximated as $1 + x\delta t$.
2. The volatility $\sigma^2\delta t$ remains the same under real world probabilities $\{q, q - q\}$ and risk neutral probabilities $\{p, 1 - p\}$. We denote the risk free interest rate as r

Then, from the real world we, have:-

$$\begin{aligned}
S_0 e^{\mu\delta t} &= S_0(1 + \mu\delta t) = q(S_0U) + (1 - q)S_0D \\
= e^{\mu\delta t} &= 1 + \mu\delta t = q(U) + (1 - q)D \tag{1} \\
\sigma^2\delta t &= qU^2 + (1 - q)D^2 - (qU + (1 - q)D)^2 \\
= \sigma^2\delta t &= qU^2 + (1 - q)D^2 - (e^{\mu\delta t})^2 \\
= \sigma^2\delta t &= qU^2 + (1 - q)D^2 - e^{2\mu\delta t} \\
= \sigma^2\delta t &= qU^2 + (1 - q)D^2 - e^{2\mu\delta t} \\
= \sigma^2\delta t &= qU^2 + (1 - q)D^2 - (1 + 2\mu\delta t) \tag{2}
\end{aligned}$$

We get 2 more equations from the risk neutral probabilities. They are as follows:-

$$= e^{\mu\delta t} = 1 + r\delta t = p(U) + (1 - r)D \tag{3}$$

$$= \sigma^2 \delta t = pU^2 + (1-p)D^2 - (1+2r\delta t) \quad (4)$$

We subtract (4) from (2) to get the following:-

$$\begin{aligned} 0 &= U^2(q-p) + D^2(1-q-(1-p)) - ((1+2\mu\delta t) - (1+2r\delta t)) \\ 0 &= (q-p)U^2 - (q-p)D^2 - 2(\mu-r)\delta t \\ 0 &= (q-p)(U^2 - D^2) - 2(\mu-r)\delta t \\ 0 &= (q-p)(U-D)(U+D) - 2(\mu-r)\delta t \end{aligned} \quad (5)$$

If we subtract (3) from (1) we get:-

$$\begin{aligned} (\mu-r)\delta t &= qU + (1-q)D - pU - (1-p)D \\ (\mu-r)\delta t &= (q-p)(U-D) \end{aligned} \quad (6)$$

substituting $(\mu-r)\delta t$ from (6) in (5), we get:-

$$\begin{aligned} 0 &= (q-p)(U-D)(U+D) - 2(q-p)(U-D) \\ 0 &= (q-p)(U-D)(U+D-2) \end{aligned} \quad (7)$$

In general $q \neq p$ and $u \neq D$. Hence, for (7) to be true, we must have

$$U + D = 2 \quad (8)$$

Now, we have the following 3 equations to solve for q , U and D :-

$$qU + (1-q)D = 1 + \mu\delta t \quad (9)$$

$$qU^2 + (1-q)D^2 - (1+2\mu\delta t) = \sigma^2\delta t \quad (10)$$

$$U + D = 2 \quad (11)$$

We can replace $1+2\mu\delta t$ in (10) with $2qU + 2(1-q)D - 1$ using (9). So we get :-

$$\begin{aligned} qU^2 + (1-q)D^2 - (2qU + 2(1-q)D - 1) &= \sigma^2\delta t \\ qU^2 + (1-q)D^2 - 2qU - 2(1-q)D + 1 &= \sigma^2\delta t \\ qU(U-2) + (1-q)D(D-2) + 1 &= \sigma^2\delta t \end{aligned}$$

We note that $U(U-2) = (2-D)(-D) = D(D-2)$ using (11). So, we have :-

$$\begin{aligned} qU(U-2) + (1-q)D(D-2) + 1 &= \sigma^2\delta t \\ (q + (1-q))D(D-2) + 1 &= \sigma^2\delta t \\ D^2 - 2D + 1 &= \sigma^2\delta t \\ (D-1)^2 &= \sigma^2\delta t \\ D &= 1 \pm \sqrt{\sigma^2\delta t} \end{aligned}$$

We know that D has to be less than 1 as it indicates the state when the stocks go down. Hence

$$D = 1 - \sqrt{\sigma^2 \delta t} \approx e^{-\sqrt{\sigma^2 \delta t}} \quad (12)$$

$$U = 1 + \sqrt{\sigma^2 \delta t} \approx e^{\sqrt{\sigma^2 \delta t}} \quad (13)$$

This approximation also gives $UD = 1$. We can then use (??), (12) and (13) to get q .

$$\begin{aligned} q &= \frac{1 + \mu \delta t - D}{U - D} \\ q &= \frac{1}{2} \frac{1 + \mu \delta t - D}{1 - D} \\ q &= \frac{1}{2} + \frac{\mu \delta t}{2(1 - D)} \\ q &= \frac{1}{2} + \frac{\mu \delta t}{2\sqrt{\sigma^2 \delta t}} \end{aligned} \quad (14)$$

Hence, we observe that U and D do not depend on μ but only σ