Introduction to Stochastic Differential Equations
MATH 6490–1 – Spring 2008
Homework 4

Due Date: Friday, May 2 at 5 PM

This homework has 225 points available but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. If you score more than 100 points, the extra points do count toward your homework total.

1 Theoretical Calculations

1.1 Thermal Forcing with Nonlinear Confining Potential (15 points)

Consider a microscale particle moving through a fluid subject to three forces: an external applied confining force, friction, and random forces due to interactions (such as collisions) with the fluid molecules. A standard way of describing the dynamics of such a particle (along a given direction) is through the Ito stochastic differential system:

$$\mathrm{d}X = V\,\mathrm{d}t, \tag{1a}$$

$$m\,\mathrm{d}V = \left[-\frac{\mathrm{d}U}{\mathrm{d}x}(X) - \gamma V\right]\mathrm{d}t + \sigma(X,V)\,\mathrm{d}W(t), \tag{1b}$$

with initial data

$$X(t=0) = X_0, \qquad V(t=0) = V_0. \tag{1c}$$

Here $X(t)$ denotes the position of the particle as a function of time t (with deterministic initial value X_0), $V(t)$ denotes the velocity of the particle (with deterministic initial value V_0), $U(x)$ is a smooth, confining potential ($\liminf_{|x|\to\infty} |U(x)|/|x| = \infty$), γ denotes a friction coefficient, and $\sigma(x,v)$ describes the strength of the thermal forcing (but does not have dimensions of force!).

The laws of equilibrium statistical physics imply that, once a system has settled down to a (statistically stationary) thermal equilibrium state, the probability density for the position and momentum should obey the Gibbs-Boltzmann distribution:

$$p_{X,V}(x,v) = \exp\left(-\frac{\frac{1}{2}mv^2 + U(x)}{k_B T}\right),$$

where $k_B = 1.38 \times 10^{-23} J/K$ is Boltzmann's constant and T is the absolute temperature. What functions $\sigma(x,v)$ would make the dynamics described by (1) consistent with this law? Provide a precise mathematical argument.

1.2 Accumulation of Reward with Growth (85 points)

Consider a function $V(s,t)$ describing the value of the reward accumulated beginning at time s and ending at time t which obeys the following dynamics in each realization:

$$\frac{\partial V(s,t)}{\partial t} = \lambda(X(t))V(s,t) + r(X(t),t),$$
$$V(s,t=s) = 0.$$

Here $X(t)$ is a solution of an Itô stochastic differential equation

$$\mathrm{d}X(t) = a(X,t)\,\mathrm{d}t + b(X,t)\mathrm{d}W(t),$$

$r(x,t)$ describes the rate at which reward is accumulated while visiting state x at time t, $\lambda(x)$ denotes a growth rate at which the already accumulated reward appreciates or depreciates in value (it depends only on the state and not explicitly on time). The functions $r(x,t)$ and $\lambda(x)$ are deterministic.

a. (**20 points**) Derive a deterministic partial differential equation or system of partial differential equations which will produce the expected value of the accumulated reward by time t, starting from a time s at which the state of the system is known to take the value x. Techniques similar to those used in class will work here, but the result is a little more complicated than those presented in class. Also, certain approaches will run into serious obstacles, so if you find yourself stuck with a mess, consider trying another one of the approaches we developed in class.

b. (**15 points**) Solve your equation explicitly for the case in which a, b, and λ are each constant.

c. (**15 points**) Compare your exact formula against an average over Monte Carlo simulations of the trajectories.

d. (**15 points**) Examine a model in which a, b, and/or c are non-constant, and solve your equations from part (a) either analytically or numerically.

e. (**10 points**) Compare your result from part (d) against an average over Monte Carlo simulations of the trajectories.

f. (**10 points**) Can your equations for the expected accumulated value be applied to a related model with "killing"?

1.3 Dying Brownian Motion (55 points)

Consider Brownian motion in a spherical annulus $D = \{\mathbf{x} \in \mathbb{R}^d : R_1 < |\mathbf{x}| < R_2\}$ which is killed at a rate $c(|\mathbf{x}|)$ depending only on the distance from the origin.

a. (**10 points**) Find an explicit deterministic equation for which the solution yields the probability that this Brownian motion process starting from $x \in D$ is killed before it reaches the boundary of D. Be sure to explain your reasoning.

b. (**5 points**) Find an explicit deterministic equation for which the solution yields the probability that this Brownian motion process starting from $x \in D$ reaches the inner boundary before either being killed or reaching the outer boundary. Be sure to explain your reasoning.

c. (**5 points**) Find an explicit deterministic equation for which the solution yields the expected time until this Brownian motion process is either killed or reaches one of the boundaries. Be sure to explain your reasoning.

d. (**20 points**) Solve the above equations either analytically or numerically for two substantially different choices of killing rate function $c(r)$. (That is, your two choices of killing rates should not be trivially related to each other as, for example, being proportional to each other.)

e. (**15 points**) Compare your results from part (d) with numerical results obtained from an ensemble of Monte Carlo simulations.

2 Mathematical Problems

2.1 Random Walk Approximation of Diffusion Process (45 points)

Consider a discrete biased random walk process $\{Y_n\}_{n=0}^{\infty}$ on the integers such that

$$Y_{n+1} = Y_n + Z_n$$

where $\{Z_n\}_{n=0}^{\infty}$ are independent, identically distributed random variables with probability distribution

$$\text{Prob}\,(Z = z) = \begin{cases} p & \text{for } z = -1, \\ q & \text{for } z = +1, \\ r & \text{for } z = 0, \\ 0 & \text{else .} \end{cases}$$

The probabilities satisfy $0 < p, q, r < 1$ and $p+q+r = 1$. We now embed this random walk as a continuous random process $X(t)$ in time by choosing a time step Δt and step length Δx so that

$$X(n\Delta t) = Y_n \Delta x,$$

with linear interpolation between these discrete time samples. We allow the probabilities p, q, and r to depend on Δx and Δt.

a. (**20 points**) How should p, q, r, Δx, and Δt be related so that the random walk will approach a diffusion process with constant drift v and constant diffusivity K in the continuum limit $\Delta x, \Delta t \downarrow 0$? You should provide a formal calculation, but do not need to be fully rigorous here.

b. (**25 points**) Prove that the interpolated random walk is, for small but finite Δx and Δt, satisfies approximately the criteria for a diffusion process by showing that the following quantities are small:

$$\lim_{t \downarrow s} \frac{1}{t-s} \int_{|y-x|>\epsilon} P_{\Delta x,\Delta t}(s, x; t, \mathrm{d}y),$$

$$\lim_{t \downarrow s} \left[v - \frac{1}{t-s} \int_{|y-x|\le\epsilon} (y-x) P_{\Delta x,\Delta t}(s, x; t, \mathrm{d}y) \right],$$

$$\lim_{t \downarrow s} \left[K - \frac{1}{2(t-s)} \int_{|y-x|\le\epsilon} (y-x)^2 P_{\Delta x,\Delta t}(s, x; t, \mathrm{d}y) \right],$$

Here $P_{\Delta x,\Delta t}(s, x; t, \mathrm{d}y)$ is the transition probability measure for the random walk approximation (as embedded as a continuous random process with time step Δt and step length Δx) and ϵ is an arbitrary but fixed positive number.

2.2 Stochastic Integrals with Non-Markov Times (15 points)

Provide a concrete example of a stochastic integral $\int_s^t f(W(u), u)\,\mathrm{d}W(u)$ and a random time τ, which is not a Markov time with respect to the filtration $\mathcal{F}^W$ generated by the underlying Wiener process, such that

$$\mathbb{E}\left[\int_s^\tau f(W(u), u)\,\mathrm{d}W(u)\right] \neq 0.$$

(Equality must hold if τ were a Markov time with $\mathbb{E}\tau < \infty$).

2.3 Asymptotic stability (10 points)

Show that the fixed point solution $X(t) \equiv 0$ of the stochastic differential equation

$$\mathrm{d}X = aX(1 + X^2)\,\mathrm{d}t + bX\,\mathrm{d}W(t)$$

is stochastically asymptotically stable if $a + \frac{1}{2}b^2 < 0$.