Introduction to Stochastic Differential Equations
MATH 6490–1 – Spring 2008
Homework 3

Due Date: Friday, May 2 at 5:00 PM

This homework has 270 points plus 35 bonus points available but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. If you score more than 100 points, the extra points do count toward your homework total.

1 Theoretical Calculations

1.1 Brownian Motion on an Ellipse (15 points + 15 bonus points)

One natural way to try to define Brownian motion on an ellipse:

$$\left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$$

with $a, b > 0$ is by a two-dimensional random process $\mathbf{X}(t) = (X_1(t), X_2(t))$ with:

$$X_1(t) = a\cos W(t), \qquad X_2(t) = b\sin W(t),$$

where $W(t)$ is a one-dimensional Wiener process.

a. (**15 points**) Find an autonomous Ito stochastic differential equation and an autonomous Stratonovich stochastic differential equation which $\mathbf{X}(t)$ satisfies. (Autonomous means the coefficients should be independent of time).

b. (**15 bonus points**) Can you suggest other ways to define Brownian motion on an ellipse?

1.2 Stratonovich solution method (15 points)

One trick to solving stochastic differential equations that works in some special cases is to express the SDE in Stratonovich form. If this has a simple form, the solution can sometimes be guessed because the chain rule in Stratonovich calculus takes the familiar form.

Consider then the Ito stochastic differential equation

$$\begin{aligned} \mathrm{d}X &= X^3\,\mathrm{d}t + X^2\,\mathrm{d}W(t), \\ X(t=0) &= X_0. \end{aligned}$$

a. (**5 points**) Convert the stochastic differential equation to Stratonovich form.

b. (**5 points**) Derive a solution to the stochastic differential equation by just solving the Stratonovich SDE using ordinary calculus methods.

c. (**5 points**) Verify that your solution satisfies the Ito stochastic differential equation by applying Ito's formula.

2 Applications

2.1 Harmonic Oscillator with Noisy Frequency (65 points plus 20 bonus points)

Consider a dynamical system which can be modelled as a harmonic oscillator with "white noise" fluctuations in the frequency as well as an independent "white noise" forcing:

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = -\left(\omega_0^2 + a\frac{\mathrm{d}W_1}{\mathrm{d}t}\right)X + b\frac{\mathrm{d}W_2}{\mathrm{d}t}, \tag{1a}$$

where $W_1(t)$ and $W_2(t)$ are independent Wiener processes, and the initial data is deterministic:

$$X(t=0) = X_0, \qquad \frac{\mathrm{d}X}{\mathrm{d}t}(t=0) = V_0. \tag{1b}$$

The natural frequency of the harmonic oscillator is ω_0, whereas a measures the strength of the noisy fluctuations of the frequency and b measures the strength of the external forcing.

a. (**5 points**) Criticize at least one aspect of this model.

b. (**15 points**) Rewrite this equation in proper stochastic differential form. Find the exact solution for the special case $a = 0$.

c. (**20 bonus points**) Find the exact solution for the case $a \neq 0$.

d. (**15 points**) Calculate the mean and variance of $X(t)$.

e. (**10 points**) Simulate the stochastic harmonic oscillator (1) using the Milstein method; note that one need only add a Milstein correction term corresponding to the noise source $W_1(t)$ because the other has constant coefficient. For $a = 0$, plot a realization of the exact solution together with a numerical approximation, using a time step large enough that the difference between the solutions can be seen.

f. (**10 points**) Provide a quantitative illustration that the Milstein method does indeed provide a strong first order accurate solution (such as adapting the convergence study in `milstrong.m`) for this stochastic differential equation, both for $a = 0$ and one case with $a \neq 0$. When you don't have an exact solution, you can still check the order of convergence by taking your finest resolution simulation as a surrogate for the exact solution, and see how quickly the coarser simulations converge to the results of the fine scale simulation.

g. (**10 points**) Provide a quantitative illustration that both the Milstein method and the Euler-Marayama method do indeed provide weak first order accurate approximations to the solution (for $a = 0$ and one case where $a \neq 0$) through consideration of numerical simulations of the mean and variance of $X(t)$. You may wish to adapt the convergence study in `emweak.m` for this purpose.

2.2 Thermal Fluctuations with Constant External Force (80 points)

Consider a microscale particle moving through a fluid subject to three forces: an external constant applied (electrical, gravitational, etc.) field, friction, and random forces due to interactions (such as collisions) with the fluid molecules. A standard way of describing the dynamics of such a particle (along a given direction) is through the stochastic differential system:

$$\mathrm{d}X = V\,\mathrm{d}t, \tag{2a}$$

$$m\,\mathrm{d}V = (f - \gamma V)\,\mathrm{d}t + \sqrt{2k_B T\gamma}\,\mathrm{d}W(t), \tag{2b}$$

with initial data

$$X(t=0)=X_0, \qquad V(t=0)=V_0. \tag{2c}$$

Here $X(t)$ denotes the position of the particle as a function of time t (with deterministic initial value X_0), $V(t)$ denotes the velocity of the particle (with deterministic initial value V_0), f is a constant externally applied force, γ denotes a friction coefficient, k_B is Boltzmann's constant, and T is the absolute temperature.

a. (**15 points**) Calculate the mean and variance of the velocity $V(t)$ and position $X(t)$ of the particle as a function of time. How do your results compare to the case of no external force $f=0$?

b. (**10 points**) Calculate the correlation function of $V(t)$ and compare it to the case of no external force $f=0$.

c. (**10 points**) Generalize your answers to the case where the initial position X_0 and velocity V_0 are themselves random variables. You may assume these initial values are independent of the thermal noise at times $t>0$, but you should not assume X_0 and V_0 are independent.

d. (**10 points**) Simulate the equation (2) numerically using the Euler-Marayama method plot a realization of the exact solution together with a strong numerical approximation.

e. (**5 points**) Comment on how the physics of the equation (2) are manifested in the exact solution and plots.

f. (**10 points**) Examine the strong order of accuracy of the Euler-Marayama method for this equation (such as by adapting `emsstrong.m`) and comment on your findings.

g. (**10 points**) Develop a numerical approximation for the correlation function of $V(t)$ using your numerical simulations (not the exact solution).

h. (**10 points**) Examine how the accuracy of this approximation of the correlation function depends on the time step, and comment on your findings.

2.3 Long-Time Behavior of Stochastic Exponential Population Growth (25 points)

In class, we showed that solutions of the stochastic differential equation representing a model of stochastic population growth

$$\mathrm{d}X = aX\,\mathrm{d}t + bX\,\mathrm{d}W(t), \qquad X(t=0)=X_0$$

with constants a and b satisfying $0 < a < \frac{1}{2}b^2$ had the following properties:

- the population would eventually die out: $\lim_{t\to\infty} X(t) = 0$,
- the mean population would grow exponentially rapidly: $\lim_{t\to\infty}\langle X(t)\rangle = \langle X_0\rangle \mathrm{e}^{at}$.

Conduct numerical simulations over a sequence of long time intervals which illustrate these apparently paradoxical limiting behaviors.

3 Mathematical Problems

3.1 Multidimensional Martingale functions of Brownian motion (15 points)

Let $\mathbf{W}(t)$ be a d-dimensional Wiener process with $d \geq 2$; each component is an independent Wiener process. Find a nonlinear scalar function $f(\mathbf{x})$ such that $f(\mathbf{W}(t))$ is a martingale which is invariant under rotations.

3.2 Ito Formula Modifications for Non-Smooth Functions (55 points)

a. (**10 points**) Suppose Ito's formula (in integrated form) were to be applied naively to express $|W(t)|$ in terms of a stochastic integral and ordinary integral over the time interval $[0, t]$. For the moment, ignore the fact that the function $|x|$ is not smooth at the origin; it is otherwise piecewise smooth so that one can make sense out of Ito's formula. Unfortunately, the answer you get in this way is wrong. Show in a precise and concrete way how one can see that $|W(t)|$ can't be equal to the integrated expression obtained from a naive application of Ito's formula.

b. (**10 points**) The reason for this discrepancy is that Ito's formula does not work properly for functions f of a stochastic process $X(t)$ when f fails to be continuously differentiable, even at isolated points. One can show, however, that Ito's formula is correct when used to express $f(X(t))$ in terms of stochastic and ordinary integrals provided f is:

  - continuously differentiable *everywhere*, and
  - twice continuously differentiable everywhere except possibly at a finite set of isolated points.

You may take this fact as given; there is no need for you to prove it (it just involves some technical approximation lemmas).

Construct a function $f_\epsilon(x)$ which satisfies these criteria and agrees with the function $|x|$ everywhere except for a small interval $x \in (-\epsilon, \epsilon)$ about the origin. Choose as simple a modification as possible to make the rest of the problem more managable. Now apply Ito's formula to express $f_\epsilon(W(t))$ in terms of a stochastic integral and ordinary integral over the interval $[0, t]$.

c. (**15 points**) Examine now what happens to your relation for $f_\epsilon(W(t))$ as ϵ approaches zero. Note that $\lim_{\epsilon \downarrow 0} f_\epsilon(W(t)) = |W(t)|$ pointwise, otherwise you did something wrong. On the right hand side of your relation, the stochastic integral will have a nice $\epsilon \downarrow 0$ limit; show this rigorously. Simplify the ordinary integral as much as possible in the $\epsilon \downarrow 0$ limit, but you will probably find that you cannot completely evaluate the limit. Therefore, after the simplifications, you should have that $|W(t)|$ can be expressed as the sum of a nice stochastic integral plus an unusual term involving a certain limit.

d. (**5 points**) Compare your result with the naive relation you wrote down in (a). Explain why the new term you derived in the relation could plausibly fix the problem with that relation which you pointed out.

e. (**15 points**) Develop a numerical simulation which shows (in a suitably approximate way) that the relation you derived in (c) is quantitatively correct.