Introduction to Stochastic Differential Equations
MATH 6490–1 – Spring 2008
Homework 2

Due Date: Friday, May 2 at 5:00 PM

This homework has 145 points available but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. If you score more than 100 points, the extra points do count toward your homework total.

1 Theoretical Calculations

1.1 Connections between Conditional Expectation and Correlations (15 points)

Suppose that the conditional expectation of random variable Y is a linear function of the conditioning random variable X:

$$\mathbb{E}[Y|X] = aX + b.$$

Express the constants a and b in terms of the means, variances, and correlation coefficient of the random variables X and Y.

1.2 Covariance of Random Sums (20 points)

Suppose that $\{X_n\}_{n=1}^{\infty}$ is a collection of independent, identically distributed random variables, each with mean μ_X and standard deviation σ_X, and that N_1 and N_2 are independent, identically distributed geometric random variables which are also independent of the $\{X_n\}_{n=1}^{\infty}$:

$$p_{N_i}(j) = \text{Prob}\,(N_i = j) = p(1-p)^j \text{ for } i = 1, 2 \text{ and } j = 0, 1, 2, \ldots$$

and define the sums:

$$S_N \equiv \sum_{n=1}^{N} X_n,$$

with $S_0 = 0$. Calculate the covariance of the sums involving random numbers N_1 and N_2 of terms:

$$\text{Cov}\,(S_{N_1}, S_{N_2})\,.$$

Statistics of random sums are relevant in applications where the effects of an uncertainty are accumulated over an uncertain period of time (or iterations). Often in applications the random variables N_1 and N_2 which terminate the sum actually depend on the values of the random variables $\{X_n\}_{n=1}^{\infty}$, but we require more sophisticated martingale techniques to handle those cases.

1.3 Integrals Involving Wiener Process (20 points)

Calculate the expectation $\mu_j(t) \equiv \mathbb{E} Y_j(t)$ and correlation function (note the shift in usual probabilistic terminology)

$$C_j(t, t') = \mathbb{E}\left[(Y_j(t) - \mu_j(t))(Y_j(t') - \mu_j(t'))\right]$$

of the random functions $Y_j(t)$ defined by the following integrals involving the Wiener process $W(t)$:

$$Y_1(t) = \int_0^t W^3(s)\,\mathrm{d}s,$$

$$Y_2(t) = \int_0^t W^3(s)\,\mathrm{d}W(s),$$

$$Y_3(t) = \int_0^t s^3\,\mathrm{d}W(s)$$

1.4 Martingalization of Wiener Integral (15 points)

Find a deterministic function $f(t)$ such that

$$\int_0^t sW(s)\,\mathrm{d}s - f(t)W(t)$$

is a martingale. This may seem like an idle game but the construction and identification of martingales out of given ingredients is a fundamental strategy in more advanced stochastic analysis.

2 Numerical Computations

2.1 Discrete Time Approximation of Wiener Process (40 points)

Write a computer program which generates a discrete-time approximation to the Wiener process by approximating it with a random walk.

a. (**15 points**) Specifically, divide time into small intervals of width Δt, and simulate (in a precise way) the increments the Wiener process undergoes over each interval. That is, you should be accurately simulating the statistics of the Wiener process on the discrete points $\{j\Delta t\}$ for integer values of j. Plot a continuous-time approximation of the Wiener process by linearly interpolating between the values computed at the points $\{j\Delta t\}$. Provide a few example plots with different values of Δt.

b. (**15 points**) Discuss to what extent your random walk approximation has the properties defining the Wiener process. Which of these properties are exactly true for your random walk approximation, and which are only approximately true? Show that the approximate properties become exact in the limit $\Delta t \downarrow 0$. Support your statements with careful mathematical calculations.

c. (**5 points**) Calculate the total variation of your random walk approximations over a unit interval, and describe how it behaves as Δt is reduced. How do your observations relate to one property of the Wiener process?

3 Mathematical Problems

3.1 Conditional Covariance Formula (15 points)

Define

$$\mathrm{Cov}\,(X, Y|\mathcal{G}) = \mathbb{E}[(X - \mu_{X|\mathcal{G}})(Y - \mu_{Y|\mathcal{G}})|\mathcal{G}],$$

where $\mu_{X|\mathcal{G}} = \mathbb{E}(X|\mathcal{G})$ and $\mu_{Y|\mathcal{G}} = \mathbb{E}(Y|\mathcal{G})$ and $\mathcal{G}$ is a sub σ-algebra of the full σ algebra of measurable sets on the probability space. Derive a formula for $\mathrm{Cov}\,(X, Y)$ in terms of the conditional expectations $\mathbb{E}(X|\mathcal{G})$ and $\mathbb{E}(Y|\mathcal{G})$ as well as the conditional covariance $\mathrm{Cov}\,(X, Y|\mathcal{G})$.

3.2 Variance of a Random Integral (20 points)

a. (**5 points**) Show directly (without any discretization) that, for any continuous random function $X(t)$ and deterministic function $f(t)$

$$\begin{aligned}
\mathbb{E}\left[\int_0^t f(t')X(t')\,\mathrm{d}t'\right] &= \int_0^t f(t')\left[\mathbb{E}X(t')\right]\,\mathrm{d}t', \\
\operatorname{Var}\left[\int_0^t f(t')X(t')\,\mathrm{d}t'\right] &= \int_0^t \int_0^t f(t')f(t'')\operatorname{Cov}\left(X(t'), X(t'')\right)\mathrm{d}t'\,\mathrm{d}t'' \qquad (1)
\end{aligned}$$

b. (**10 points**) Compare these formulas with those corresponding to the case of a finite, discrete collection of random variables $\{X_j\}_{j=1}^N$:

$$\begin{aligned}
\mathbb{E}\left(\sum_{j=1}^N c_j X_j\right) &= \sum_{j=1}^N c_j \mathbb{E}X_j, \\
\operatorname{Var}\left(\sum_{j=1}^N X_j\right) &= \sum_{j=1}^N c_j^2 \operatorname{Var} X_j + \sum_{j=1}^N \sum_{\substack{j'=1 \\ j'\neq j}}^N c_j c_{j'} \operatorname{Cov}\left(X_j, X_{j'}\right).
\end{aligned}$$

where $\{c_j\}_{j=1}^N$ are deterministic constants. Show in particular that applying this formula to Riemann sum approximations to a random integral and passing to the limit of an infinitely fine partition recovers formula (1). Why does the formula for the variance of the integral not include an analogue to the sum of variances nor exclude the evaluation of the covariance at equal times?

3.3 Inverse Rescaled Wiener Process (15 points)

Show in a precise way how $tW(1/t)$ has the properties of a Wiener process if $W(t)$ is a Wiener process.