

# Subdiffusion and Superdiffusion in Lagrangian Stochastic Models of Oceanic Transport

Emilio Castronovo\* and Peter R. Kramer

March 11, 2004

## Abstract

To better understand the capacity of Lagrangian Stochastic Models to simulate superdiffusive and subdiffusive tracer motion in oceanic turbulence, we examine their performance on a simple class of model velocity fields which support subdiffusive or superdiffusive regimes of tracer transport associated to power-law regions of the Lagrangian power spectrum. We focus on how well the Lagrangian Stochastic Models can replicate the subdiffusion and superdiffusion in these models, when they are provided with exact Lagrangian information. This simple test reveals fundamental limitations in the type of subdiffusion and superdiffusion which a standard hierarchy of Lagrangian Stochastic Models is able to quantitatively approximate.

## 1 Introduction

A key goal in applied scientific systems involving turbulent fluids is the practical ability to predict the transport properties for substances and objects immersed in the fluid over moderate to long periods of time. Examples include moisture, pollutants, and chemicals in the atmosphere, salinity and contaminants in the ocean, sediment in rivers, and reactants and soot in mixing reactors. Detailed direct numerical simulations [23] of the fluid equations coupled to the dynamics of the immersed substances are computationally very expensive and in many cases impossible on even current supercomputers for turbulent flows at high Reynolds number [12]. For efficient and practical predictions, simpler and more manageable models are sought which capture the important phenomena of interest without needing to calculate all the details of the complex turbulent system. The Lagrangian Stochastic Model (LSM) framework is a promising approach under active development [14, 15, 19, 20]. The

---

\*Corresponding author: [castre@rpi.edu](mailto:castre@rpi.edu). Mailing address and affiliation for both authors: Department of Mathematical Sciences, Rensselaer Polytechnic Institute, 301 Amos Eaton Hall, 110 8th Street, Troy, NY 12180

basic idea is to model a representative (Lagrangian) particle of the immersed substance by a low-dimensional system of equations, with the influence of the turbulent environment represented in terms of parameterized coefficients and random noise. We will concentrate in the present work on the application of LSMs to the prediction of the transport of passive tracers in random flows with specified statistics. That is, we put aside the problem of predicting the turbulent flow, though LSMs are used for this as well [14, 15], and restrict our attention to the “passive scalar field” problem in which the presence of the immersed substance does not appreciably influence the fluid flow. This assumption is reasonable for substances composed of small, light particles, and the understanding of the transport of passive scalar fields will certainly inform the development of models for more complex couplings between the fluid and immersed substances.

The development and criticism of Lagrangian Stochastic Models has emphasized physical scaling and ordering principles [10, 14, 19], some mathematical asymptotic analysis of the predictions of the LSMs [14], and comparison with sample data obtained by direct numerical simulation of turbulent flows (with Reynolds number typically considerably reduced below the realistic value so as to make the computation feasible) [3, 4, 23]. We offer here another approach to analyzing LSMs using turbulent velocity field models which are simple enough to permit precise mathematical and asymptotic analysis and rich enough to exhibit some features of the turbulent system of interest. These mathematical models are of intermediate complexity in that they still entail a detailed representation of the velocity field, but use a convenient statistical description rather than detailed data. The LSM methodology can still be meaningfully applied to these mathematical model flows, and the predictions compared with exact results. One advantage of using mathematical models is that statistics of passive tracers can be represented as analytical formulas for whole classes of flows, and the generic features can be extracted through asymptotic and other techniques. The general insight so obtained can be used in the formulation of improved models. Eulerian closure approximations for turbulence modeling, such as the direct interaction approximation and those based on the renormalization group and renormalized perturbation theory, have been previously studied by this mathematical modeling approach [1, 9].

Lagrangian Stochastic Modeling involves two important modeling steps. First of all, the structure of the LSM must be decided [14]. We will work within the framework used by Berloff and McWilliams [4] restricted to the case of statistically homogenous and stationary flows, for which the LSMs take the form of linear, continuous autoregressive (CAR) Markov processes [22]. The next main decisions to be made are the order of the CAR process to be used and the means of parameterizing the coefficients. One seeks a LSM with enough parameters to replicate the desired physical phenomena but not so many as to excessively complicate the model for little or ambiguous gain in modeling power [4, 14, 21]. Once the structure of the LSM is determined, many of its coefficients can fairly confidently be linked to certain Lagrangian statistics of the tracer. One generally would like to express the parameters in terms of an Eulerian description of the flow from a fixed observation frame, but this Eulerian to Lagrangian transformation is generally very difficult and fully satisfactory procedures are still not available [4]. While we plan to address this issue of Eulerian to Lagrangian parameterization in the future, we do not attempt this task in the present paper.

Rather, we focus on the more tractable aspect of Lagrangian Stochastic Modeling, which is the choice of the structure of the LSM adequate for the turbulent transport predictions in the application of interest. When the subgrid scales of the velocity can at least locally be described by the Kolmogorov theory for isotropic turbulence [11, 16], physical theory suggests a first order framework is appropriate [14, 21], though a second order refinement has also been suggested [17]. But turbulent flows in contexts such as atmospheric and oceanographic modeling may not be well described on all relevant subgrid scales by the Kolmogorov theory, due to the presence of waves and coherent structures [3, 11]. The LSMs postulated for oceanic transport [4, 18] therefore have a different character than those used for Kolmogorov subgrid scale turbulence. Berloff and McWilliams [4] chose the structure for their oceanographic transport models phenomenologically, finding that they require a second order LSM to replicate subdiffusive transport and a third order LSM (including “hyperacceleration” as a state variable) to replicate superdiffusive transport occurring in certain spatial regions in their double-gyre numerical simulations [3].

Our aim in the present work is to focus on this modeling principle through isolating it mathematically in a simpler context. We will consider a tracer in a statistically homogeneous and stationary flow, so that its Lagrangian statistics will be statistically stationary, and then directly specify in Section 2 a certain spectral structure for the correlation function of the *Lagrangian* tracer velocity. These model spectra for the random tracer velocity are adapted from previous work [2, 7, 13], where they were employed for analyses of other turbulent diffusion phenomena in a similar spirit. The specified structure of the Lagrangian velocity correlation function is broad enough to comprise regimes of subdiffusive, diffusive, and superdiffusive tracer behavior. While not completely general, our mathematical model is generic in the sense that the anomalous diffusion is linked to a range of wavenumbers over which the (Lagrangian) power spectrum has a power law scaling, with the exponent appearing as a parameter in our mathematical model. We will use these model Lagrangian spectra in Section 3 to explore how well the LSMs are able to replicate the associated statistical tracer dynamics. This is a rather mild test: the parameters in the LSMs will be determined using perfect Lagrangian information, and the mathematical model spectra have a rather simple and standard structure. The issue of inferring Lagrangian parameters from Eulerian data does not arise in our analysis. The reason for us posing this simple test is to determine the fundamental capacity for second and third-order LSMs of the type used in [3] for replicating subdiffusive and superdiffusive transport in a fairly generic context. Any discrepancies uncovered are due entirely to the structure of the LSM model, and not due to the complexity of a given flow, lack of homogeneity, or difficulty of converting Eulerian to Lagrangian information. By understanding the capacity of LSMs under favorable circumstances, we hope to see more clearly how one might use or improve them for more complex situations.

We will compare the evolution of the mean-square tracer displacement  $\sigma_X^2(t) \equiv \langle (X(t) - X(0))^2 \rangle$  and the time-dependent rate of diffusion  $D(t) \equiv \frac{1}{2} d\sigma_X^2(t)/dt$  for the mathematical models and the LSM approximations through asymptotic analysis and numerical plots of the explicit formulas. We find in this way that the second and third-order LSMs suffer fundamental limitations in their ability to track subdiffusion and superdiffusion arising from

Lagrangian power spectra with power-law scaling. The implications of these results and some directions for future research will be discussed in Section 4.

## 2 Model Lagrangian Spectra

We will consider how well Lagrangian Stochastic Models can replicate the dynamics of a tracer with mean zero, statistically stationary Lagrangian velocity  $v^{(L)}$ , and Lagrangian velocity correlation function given by the following model [2, 7, 9, 13]

$$\begin{aligned} R_U^{(L)}(t) &\equiv \langle v^{(L)}(t'+t)v^{(L)}(t') \rangle = \int_{-\infty}^{\infty} e^{2\pi i \omega t} E^{(L)}(|\omega|) d\omega, \\ E^{(L)}(\omega) &\equiv A_E |\omega|^{-\beta} \psi_0(|\omega|\tau_c) \psi_{\infty}(|\omega|). \end{aligned} \quad (1)$$

We work with one-dimensional Lagrangian velocity fields for simplicity; these may be viewed as any component of a multi-dimensional Lagrangian velocity field. Since we will only be presenting second order statistics for a single tracer, we do not include a particle label in the Lagrangian velocity and do not need to specify the higher order statistics (Gaussian or not). The function  $\psi_{\infty}$  is a smooth, rapidly decaying ‘‘ultraviolet cutoff’’ function on the positive real axis, with  $\psi_{\infty}(0) = 1$  (we take  $\psi_{\infty}(\omega) = e^{-\omega^2}$  in our numerical plots). We also introduce an ‘‘infrared cutoff’’ function  $\psi_0$  with the form  $\psi_0(\varpi) = (\frac{\varpi}{1+\varpi})^{\beta}$  to give the Lagrangian power spectrum the physical property that  $E^{(L)}(0)$  is a nonzero, finite number. If we then choose  $\tau_c$  as a large number, the Lagrangian power spectrum will satisfy a power scaling  $E^{(L)}(\omega) \sim A_E |\omega|^{-\beta}$  over the range of frequencies  $\tau_c \ll \omega \ll 1$ . We will consider exponents  $-1 < \beta < 1$ .

By revisiting the asymptotic calculations for the case without infrared cutoff [13], we find that the mean-square tracer displacement associated to the Lagrangian power spectrum described in (1) has the following asymptotic behavior:

$$\sigma_X^2(t) \sim \begin{cases} \sigma_U^2 t & \text{for } t \ll 1, \\ \frac{2}{1+\beta} K_{\beta} t^{1+\beta} & \text{for } 1 \ll t \ll \tau_c, \\ 2\sigma_U^2 \tau_L t & \text{for } t \gg \tau_c, \end{cases} \quad (2)$$

where  $\sigma_U^2 = \langle v^2 \rangle = R_U^{(L)}(0)$  is the mean-square (Lagrangian or Eulerian) velocity,  $\tau_L = \sigma_U^{-2} \int_0^{\infty} R_U^{(L)}(t) dt$  is the Lagrangian correlation time (which scales as  $\tau_c^{\beta}$  for large  $\tau_c$ ), and  $K_{\beta} = \frac{1}{2} A_E \pi^{\beta-(1/2)} \Gamma((1-\beta)/2) / \Gamma((2+\beta)/2)$ . For short time, the tracer motion is ballistic, and at long time, the tracer motion proceeds according to an ordinary diffusion law, as is generically the case [16]. But over the intermediate time interval  $1 \ll t \ll \tau_c$ , the tracer transport is superdiffusive when  $0 < \beta < 1$ , diffusive when  $\beta = 0$ , and subdiffusive when  $-1 < \beta < 0$ . Our model Lagrangian power spectra (1) then at least qualitatively contain the features which motivated the development of the second and third order CAR models by Berloff and McWilliams [4]. We will investigate how the LSM predictions compare with the exact results for tracers with these Lagrangian power spectra to see whether the LSMs are

capable of simulating subdiffusion and superdiffusion in a broader context. We do not claim that the spectra (1) are the most general ones exhibiting superdiffusion and subdiffusion, but they are simple and rather generic in the sense that the spectra of turbulent flows often have self-similar power law regimes, even when Kolmogorov theory does not apply [11, 12].

### 3 Comparison of Lagrangian Stochastic Models Against Exact Results

We examine next how well the LSMs developed in [4] are able to replicate the exact tracer transport for situations where the Lagrangian power spectrum is of our mathematical model form (1), under the ideal circumstances in which perfect Lagrangian information is provided to the LSM. Any discrepancies we find are a property of the structure of the LSM, and are independent of the choice of how the Lagrangian parameters of the LSM are to be calculated from the Eulerian data of the flow. We will consider the LSMs called “Markov-1,” “Markov-2,” and “Markov-3” by Berloff and McWilliams [4], according to their increasing complexity. (The “Markov-0” model corresponds to the modeling the transport by a simple eddy diffusion.) Our presentation of the LSMs will be here restricted to the case of a one-dimensional, statistically stationary velocity field; their more general form is described in [4].

Numerical comparisons of the LSM predictions for the tracer transport with the exact formulas corresponding to the model spectra (1) are presented for a superdiffusive case in Figure 1, an ordinary diffusive case in Figure 2, and a subdiffusive case in Figure 3. Our discussion in the text will be organized by increasing complexity of the LSM.

#### 3.1 Markov-1 Model

The state vector in the Markov-1 model consists of the position  $X(t)$  and velocity  $U(t)$  of the Lagrangian tracer, which are modeled by the first-order autoregressive Markov process:

$$\begin{aligned} dX(t) &= U(t) dt \\ dU(t) &= -\theta_U^{-1} U(t) dt + b_U dW(t), \end{aligned}$$

where  $W(t)$  is a standard Itô stochastic differential [8]. The parameters are determined from the Lagrangian velocity statistics through  $\theta_U = \tau_L$  and  $b_U = (2\sigma_U^2/\tau_L)^{1/2}$ , where  $\sigma_U^2 = \langle U^2(t) \rangle = R_U^{(L)}(0)$  is the variance of the Lagrangian velocity and  $\tau_L = \sigma_U^{-2} \int_0^\infty R_U^{(L)}(t) dt$  is the Lagrangian correlation time. The Markov-1 model improves upon the standard eddy-diffusion model by allowing for an initial ballistic phase in the evolution of the mean-square Lagrangian tracer displacement ( $\sigma_X^2(t) \sim \sigma_U^2 t^2$  for  $t \ll \tau_L$ ), before the eventually diffusive growth ( $\sigma_X^2(t) \sim 2\sigma_U^2 \tau_L t$  for  $t \gg \tau_L$ ). Clearly, the Markov-1 model is incapable of exhibiting any anomalous diffusion.

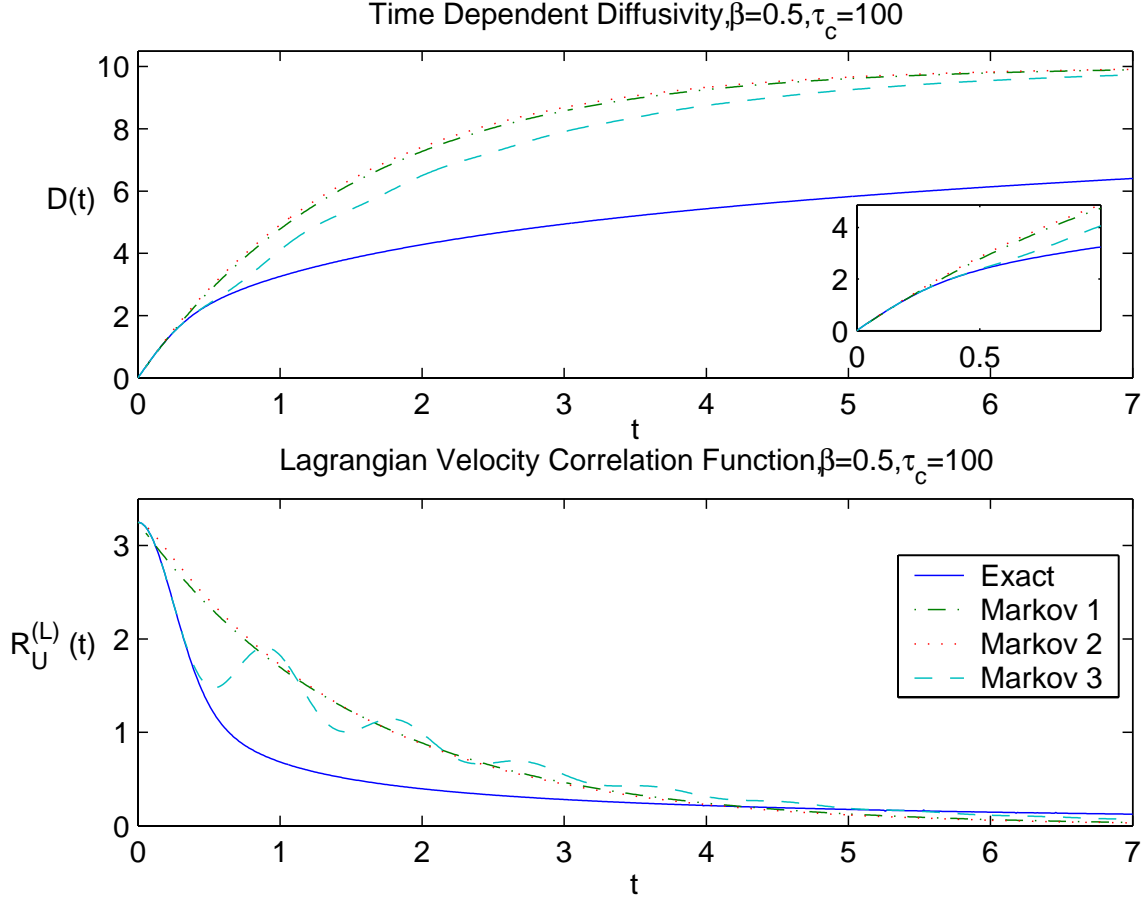


Figure 1: Comparison of time-dependent diffusivities (top panel) for the LSMs and the exact results for a model Lagrangian power spectrum (1) with  $\beta = 0.5$  and  $\tau_c = 100$ , corresponding to superdiffusive transport. The Lagrangian parameter values are  $\tau_L = 1.5$ ,  $\sigma_U = 1.8$ ,  $\sigma_A = 6.0$ ,  $\sigma_{\dot{A}} = 42$ , and  $\sigma_H = 37$ . The Markov-1 and Markov-2 approximations are nearly identical, and the Markov-3 approximation is only a slight improvement, mainly at rather short times (see inset). The time scale associated to the transition between the initial ballistic regime and the second ballistic regime in the Markov-3 approximation is  $\tau_b = 0.12$ . The lower panel shows the approximations to the Lagrangian velocity correlation function; notice the absence of negative lobes.

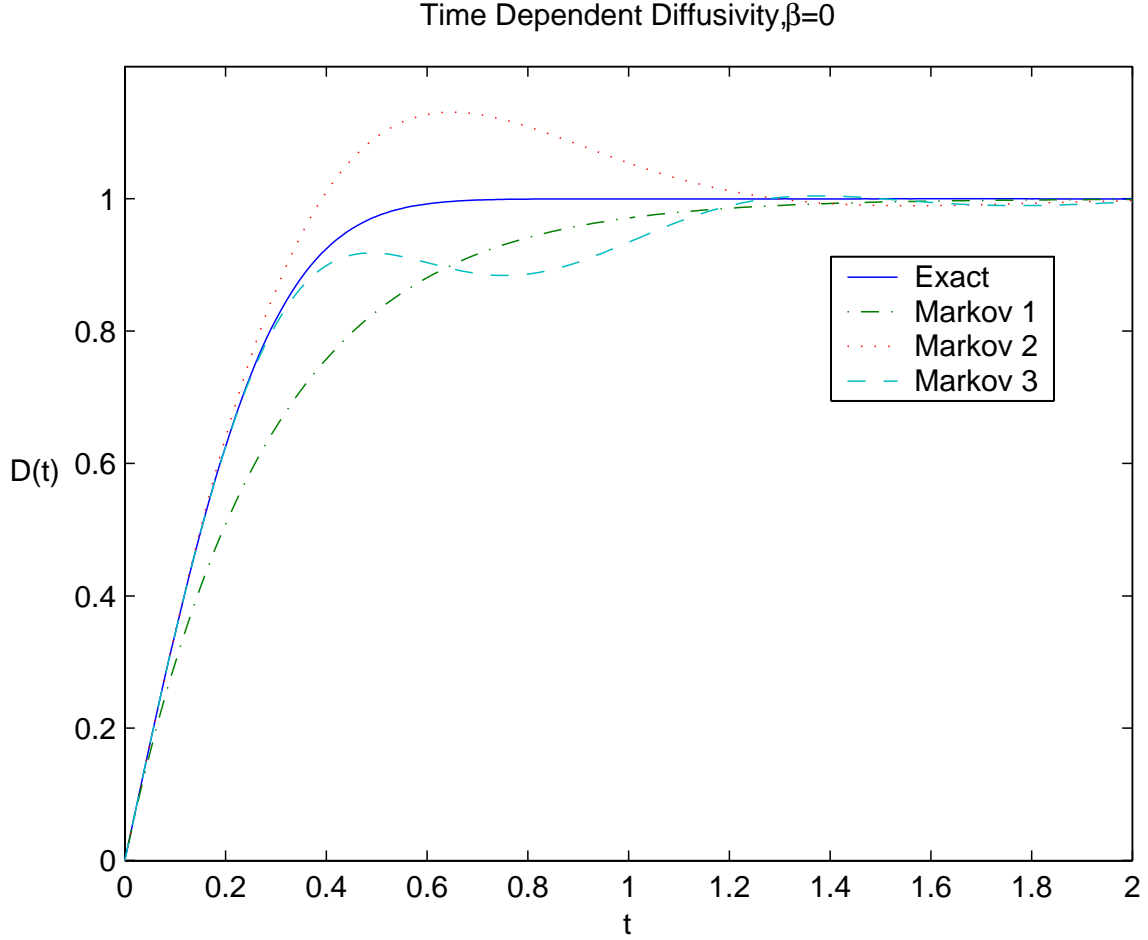


Figure 2: Comparison of time-dependent diffusivities for the LSMs and the exact results for a model Lagrangian power spectrum (1) with  $\beta = 0$  (cutoff has trivial effect), corresponding to an ordinary diffusive rate of transport. The Lagrangian parameter values are  $\tau_L = 0.28$ ,  $\sigma_U = 1.3$ ,  $\sigma_A = 5.9$ ,  $\sigma_{\bar{A}} = 46$ , and  $\sigma_H = 37$ . Notice that the Markov-2 and Markov-3 approximations improve accuracy at very short time, but introduce extraneous artifacts at intermediate time.

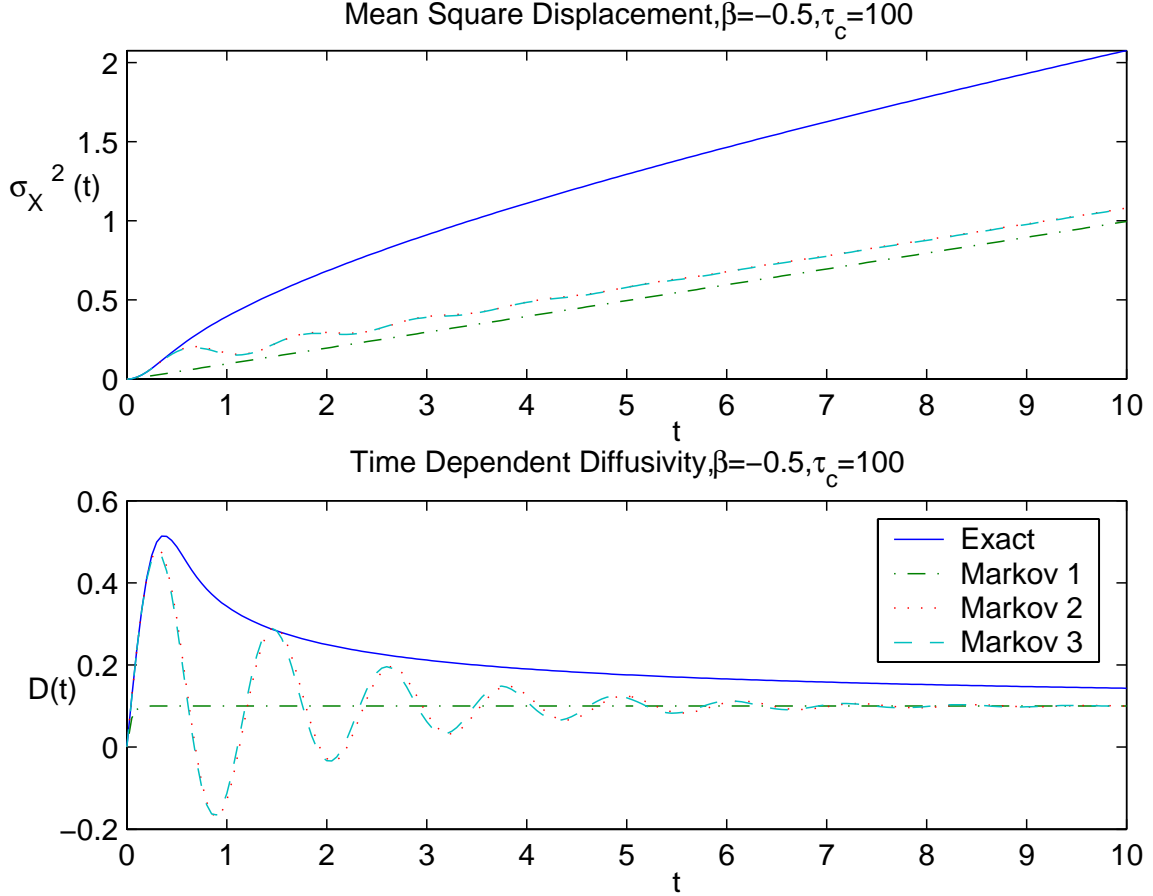


Figure 3: Comparison of mean-square displacements (top panel) and time-dependent diffusivities (bottom panel) for the LSMs and the exact results for a model Lagrangian power spectrum (1) with  $\beta = -0.5$  and  $\tau_c = 100$ , corresponding to a subdiffusive rate of transport at intermediate times. The Lagrangian parameter values are  $\tau_L = 0.040$ ,  $\sigma_U = 1.1$ ,  $\sigma_A = 6.0$ ,  $\sigma_{\dot{A}} = 50$ , and  $\sigma_H = 38$ . The Markov-2 and Markov-3 approximations are nearly identical, and improve substantially upon the Markov-1 approximation for short time, but only marginally for intermediate to long times. The oscillation time scale for the Markov-2 and Markov-3 approximations are  $\tau_{\text{osc}} = 0.18$ , and the decay time for the oscillations is  $\theta_A = 0.84$ .

## 3.2 Markov-2 Model

In the Markov-2 model, the Lagrangian particle acceleration  $A(t)$  is included in the state vector, and the tracer particle position is effectively modeled as a continuous, second-order autoregressive Markov process:

$$\begin{aligned} dX(t) &= U(t) dt, \\ dU(t) &= A(t) dt, \\ dA(t) &= -\alpha_{AU}U(t) dt - \theta_A^{-1}A(t) dt + b_A dW(t). \end{aligned}$$

The constants appearing in these equations are obtained from Lagrangian data through the relations  $\alpha_{AU} = \sigma_A^2/\sigma_U^2$ ,  $\theta_A = \sigma_U^2/(\sigma_A^2\tau_L)$ , and  $b_A = (2\sigma_A^4\tau_L/\sigma_U^2)^{1/2}$ , where  $\sigma_A^2 = \langle A^2(t) \rangle = -R_U^{(L)''}(0)$  is the mean-square Lagrangian particle acceleration.

If  $\theta_A \ll \alpha_{AU}^{-1/2}$  (or in terms of Lagrangian parameters,  $\sigma_U/(\sigma_A\tau_L) \ll 1$ ), the mean-square displacement has leading order asymptotics identical to those of the Markov-1 model; this can be seen in Figures 1 and 2 for the superdiffusive and diffusive regimes, for which the time scales have this ordering. On the other hand, when  $\theta_A \geq \frac{1}{2}\alpha_{AU}^{-1/2}$ , the simulated tracer transport exhibits oscillations with associated time scale (inverse angular frequency)  $\tau_{\text{osc}} = \left(\alpha_{AU} - \frac{1}{4\theta_A^2}\right)^{-1/2} = \left(\frac{\sigma_A^2}{\sigma_U^2} - \frac{\sigma_A^4\tau_L^2}{\sigma_U^4}\right)^{-1/2}$ . When  $\tau_{\text{osc}} \ll \theta_A$  (equivalently,  $\sigma_U/(\sigma_A\tau_L) \gg 1$ ), the short-time ballistic growth of the mean-square displacement lasts until  $t \sim \tau_{\text{osc}}$ , after which development of the mean-square displacement can be well approximated by the expression  $\sigma_X^2(t) \sim 2\sigma_U^2\tau_L [t + \theta_A (1 - e^{-t/(2\theta_A)} \cos(t/\tau_{\text{osc}}))]$  for  $t \gg \tau_{\text{osc}}$ . The Markov-2 model in this parameter regime is used to model subdiffusive transport in [4]; but we see that beyond the initial ballistic period, the Markov-2 approximation simply superposes the long-time diffusive growth with exponentially decaying oscillations about a constant mean value  $2\sigma_U^2\tau_L\theta_A$ . The damping time of the oscillations as well as the time at which the linear growth begins to dominate the mean value of the oscillatory term in the mean-square displacement are both on the order of  $\theta_A$ . The amplitude of the oscillations in the time-dependent diffusivity are a fair approximation to the amplitude of the true time-dependent diffusivity for our mathematical model in Figure 3, but the spurious oscillations cause the Markov-2 approximation for the mean-square displacement to depart substantially from the correct behavior after a quarter oscillation. The randomization of the Markov-2 model proposed by Berloff and McWilliams [5] reduces oscillations and may lead to improved performance on our example; we will explore this in future work.

## 3.3 Markov-3 Model

The Markov-3 model incorporates one further state variable called the ‘‘hyperacceleration’’  $H$ , which however is not quite the derivative of the acceleration, as inspection of the stochas-

tic model equations reveals [4]:

$$\begin{aligned} dX(t) &= U(t) dt, \\ dU(t) &= A(t) dt, \\ dA(t) &= -\alpha_{AU}U(t) dt + H(t) dt, \\ dH(t) &= -\alpha_{HA}A(t) dt - \theta_H^{-1}H(t) dt + b_H dW(t). \end{aligned}$$

The coefficients are related to Lagrangian parameters through the relations  $\alpha_{AU} = \sigma_A^2/\sigma_U^2$ ,  $\alpha_{HA} = \sigma_H^2/\sigma_A^2$ ,  $\theta_H = \sigma_A^4\tau_L/(\sigma_H^2\sigma_U^2)$ , and  $b_H = (2\sigma_U^2\sigma_H^4/(\sigma_A^4\tau_L))^{1/2}$ , where  $\sigma_H^2 = \langle H^2(t) \rangle = \sigma_A^2 - \sigma_A^4/\sigma_U^2$  is the mean-square Lagrangian hyperacceleration and  $\sigma_A^2 = \langle \dot{A}^2(t) \rangle = R_U^{(L)''''}(0)$  is the mean-square derivative of the Lagrangian acceleration.

As noted in [4], the Markov-3 model degenerates to the Markov-2 model when the decorrelation time of the hyperacceleration is very short compared to the other dynamical time scales ( $\theta_H \ll \alpha_{AU}^{-1/2}$  and  $\alpha_{HA}^{-1/2}$ , or equivalently,  $\sigma_A^5\tau_L/(\sigma_H^2\sigma_U^3) \ll 1$  and  $\sigma_A^3\tau_L/(\sigma_H\sigma_U^2) \ll 1$ ). These conditions are met for our subdiffusive model Lagrangian power spectrum, which explains why the Markov-2 and Markov-3 predictions appear almost identical in Figure 3.

The primary motivation of Berloff and McWilliams [4] in invoking the Markov-3 model was to handle superdiffusion. Superdiffusive regimes in their double-gyre simulations are associated to regimes where the Lagrangian velocity correlation function oscillates, but with more area in the positive lobes than in the negative lobes. Such Lagrangian velocity correlation functions could be generated by the Markov-3 model when the parameter values satisfy

$$\theta_H \geq \alpha_{AU}^{-1/2} \approx \alpha_{HA}^{-1/2}, \text{ or equivalently } \sigma_A^5\tau_L/(\sigma_H^2\sigma_U^3) \approx \sigma_A^3\tau_L/(\sigma_H\sigma_U^2) \geq 1. \quad (3)$$

To see whether a substantial intermediate period of superdiffusion can indeed manifest itself in the Markov-3 model, we consider the case where the inequality in these expressions is in fact strongly satisfied. The initial ballistic growth then turns over at time  $\tau_b = (\alpha_{AU} + \alpha_{HA})^{-1/2} = \sigma_A/\sigma_{\dot{A}}$  to another ballistic regime ( $\sigma_X^2(t) \sim (\sigma_H^2\sigma_U^2/\sigma_A^2)t^2$  for  $\tau_b \ll t \ll \tau_L$ ) with the coefficient reduced from that of the initial ballistic phase (since  $\sigma_H^2 < \sigma_{\dot{A}}^2$ ). If the inequality in (3) is not wide, then the remnant of this asymptotic regime could manifest itself as an apparent, short-lived superdiffusion. The numerical results in Figure 1 show, however, that any such apparent intermediate superdiffusion of the Markov-3 model does not provide a substantial improvement in approximating the true superdiffusion associated to our model Lagrangian power spectrum. We indicate in Figure 2 that the true Lagrangian velocity correlation function corresponding to this superdiffusive case is not oscillatory; the Markov-3 approximation to this correlation function introduces some spurious oscillations but remains positive. Therefore, the mechanism for superdiffusion in our model Lagrangian power spectra is not the same as that identified for the oceanic simulations in [3]. However, our asymptotic analysis does not require any assumption about how the Lagrangian velocity correlation function oscillates, and the assumptions behind it are roughly consistent with both our mathematical model as well as the superdiffusive parameter regimes described in [4], with the major objection being that the asymptotic parameter representing the separation of time scales is only approximately 2. We conclude therefore, that the superdiffusion attributed to the Markov-3 model is at best an apparent phenomenon associated to a short-lived turnover; the Markov-3 model cannot replicate superdiffusion over a long time interval.

## 4 Conclusions and Future Directions

We have found that the first three equations in a hierarchy of standard Lagrangian Stochastic Models, while able to model some forms of subdiffusive and superdiffusive transport in numerical simulations [4], do not appear well suited to simulating subdiffusion or superdiffusion in a general way, particularly when these phenomena are associated with power-law scaling regimes in the Lagrangian power spectrum.

The work presented here motivates two directions for further research. First of all, we plan to explore other simple mathematical models for the Lagrangian power spectra which are closer to those observed in the direct numerical simulations of the ocean dynamics [3]. Secondly, we are considering variations of the LSMs used by Berloff and McWilliams [4] which might be able to replicate a wider variety of subdiffusive and superdiffusive transport. One possibility is the inclusion of “spin” in the LSMs [6]. Reynolds [18] discusses how a third-order LSM is, under certain asymptotic assumptions, equivalent to a second-order LSM with spin, but these assumptions do not appear to be met for the spectra generating superdiffusive transport. Another innovation is the randomization of the parameters appearing in the LSMs, which have been found to reduce spurious oscillations in subdiffusive transport regimes [5]. Finally, LSMs with coefficients depending on the Eulerian probability density for the velocity have been shown by Reynolds [18] to produce superdiffusion with anomalous scaling. The anomalous exponent describing the scaling of the mean-square displacement is related to an exponent within the LSM describing how the coefficients depend on the Eulerian probability density for the velocity. It would be interesting to see how well the anomalous scaling produced by these LSMs compare with the exact anomalous scaling associated to the model spectra (1).

## 5 Acknowledgements

The authors would like to thank Andrew Majda for suggesting the general research program begun with this work, Jim McWilliams for a useful discussion regarding the open issues in Lagrangian stochastic modeling for oceanic transport, and Karl Sabelfeld for some helpful advice. E. C. is supported by a VIGRE graduate research fellowship.

## References

- [1] Marco Avellaneda and Andrew J. Majda. Approximate and exact renormalization theories for a model for turbulent transport. *Phys. Fluids A*, 4(1):41–56, January 1992.
- [2] Marco Avellaneda and Andrew J. Majda. Simple examples with features of renormalization for turbulent transport. *Phil. Trans. R. Soc. Lond. A*, 346:205–233, 1994.

- [3] Pavel S. Berloff and James C. McWilliams. Material transport in oceanic gyres. Part I: Phenomenology. *J. Phys. Oceanogr.*, 32:764–796, March 2002.
- [4] Pavel S. Berloff and James C. McWilliams. Material transport in oceanic gyres. Part II: Hierarchy of stochastic models. *J. Phys. Oceanogr.*, 32:797–830, March 2002.
- [5] Pavel S. Berloff and James C. McWilliams. Material transport in oceanic gyres. Part III: Randomized stochastic models. *J. Phys. Oceanogr.*, 33:1416–1445, July 2003.
- [6] Michael S. Borgas, Thomas K. Flesch, and Brian L. Sawford. Turbulent dispersion with broken reflectional symmetry. *J. Fluid Mech.*, 332:141–156, 1997.
- [7] David J. Hornthrop and Andrew J. Majda. Subtle statistical behavior in simple models for random advection-diffusion. *J. Math. Sci. Univ. Tokyo*, 1:1–48, 1994.
- [8] Peter E. Kloeden and Eckhard Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics: Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 1992.
- [9] Peter R. Kramer, Andrew J. Majda, and Eric Vanden Eijnden. Closure approximations for passive scalar turbulence: A comparative study on an exactly solvable model with complex features. *J. Statist. Phys.*, 111(3–4):565–679, May 2003.
- [10] O. Kurbanmuradov and K. Sabelfeld. Lagrangian stochastic models for turbulent dispersion in the atmospheric boundary layer. *Boundary-Layer Meteorol.*, 97:191–218, 2000.
- [11] Marcel Lesieur. *Turbulence in fluids*. Number 1 in Fluid Mechanics and its Applications. Kluwer, Dordrecht, second revised edition, 1990.
- [12] Andrew J. Majda. Real world turbulence and modern applied mathematics. In *Mathematics: frontiers and perspectives*, pages 137–151. Amer. Math. Soc., Providence, RI, 2000.
- [13] Andrew J. Majda and Peter R. Kramer. Simplified models for turbulent diffusion: Theory, numerical modelling and physical phenomena. *Phys. Rep.*, 314(4–5):237–574, June 1999.
- [14] Jean-Pierre Minier and Eric Peirano. The pdf approach to turbulent polydispersed two-phase flows. *Phys. Rep.*, 352(1-3):1–214, 2001. ISSN 0370-1573.
- [15] S. B. Pope. Lagrangian PDF methods for turbulent flows. In *Annual review of fluid mechanics*, Vol. 26, pages 23–63. Annual Reviews, Palo Alto, CA, 1994.
- [16] Stephen B. Pope. *Turbulent flows*. Cambridge University Press, Cambridge, 2000. ISBN 0-521-59886-9.
- [17] Stephen B. Pope. A stochastic Lagrangian model for acceleration in turbulent flows. *Phys. Fluids*, 14(7):2360–2375, 2002. ISSN 1070-6631.

- [18] A. M. Reynolds. On Lagrangian stochastic modelling of material transport in oceanic gyres. *Phys. D*, 172(1-4):124–138, 2002. ISSN 0167-2789.
- [19] K. K. Sabelfeld and O. Kurbanmuradov. Two-particle stochastic Eulerian-Lagrangian models of turbulent dispersion. *Math. Comput. Simulation*, 47(2-5):429–440, 1998. ISSN 0378-4754. IMACS Seminar on Monte Carlo Methods (Brussels, 1997).
- [20] Brian Sawford. Turbulent relative dispersion. In *Annual review of fluid mechanics, Vol. 33*, pages 289–317. Annual Reviews, Palo Alto, CA, 2001.
- [21] D. J. Thomson. Criteria for the selection of stochastic models of particle trajectories in turbulent flows. *J. Fluid Mech.*, 180:529–556, 1987.
- [22] A. M. Yaglom. *Correlation theory of stationary and related random functions. Volume I: basic results*, chapter 3, pages 215–221. Springer-Verlag, Berlin, 1987.
- [23] P. K. Yeung. Lagrangian investigations of turbulence. In *Annual review of fluid mechanics, Vol. 34*, pages 115–142. Annual Reviews, Palo Alto, CA, 2002.