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Method of Multiple Scales with Three Time Scales

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Some confusion appears in the literature regarding the extension of the method of multiple scales to three or more time scales. While the work of Murdock and Wang [1] correctly indicates some obstructions to such an extension in some types of problems, other work suggests that the extension should almost never be possible [2]. These pessimistic results generally follow from the imposition of additional restrictions in the calculation, which really are not necessary [3]. We will show on some simple ODE models how a systematic implementation of the method of multiple scales can succeed in correctly capturing three active time scales, though more calculation is required than one might expect from naive considerations. On the other hand, an extension of the method of averaging to three time scales produces an incorrect approximation in these problems [4].

1 General Considerations

Even in weakly nonlinear ordinary differential equations,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} &= \mathsf{L}\mathbf{y} + \epsilon \mathbf{f}(\mathbf{y}, t, \epsilon), \\ \mathbf{y}(0) &= \mathbf{y}_{\text{in}} \end{aligned}$$

where L is a diagonalizable matrix, subtleties arise in the extension of the method of multiple scales to three time scales (1, ϵ^{-1}, ϵ^{-2}). The natural way to proceed is to seek a solution of the form

$$\mathbf{y}(t) = \mathbf{y}_{MS}(t = t, \tau = \epsilon t, \sigma = \epsilon^2 t)$$

in terms of an asymptotic expansion:

$$\mathbf{y}_{MS}(t, \tau, \sigma) = \mathbf{y}_0(t, \tau, \sigma) + \epsilon \mathbf{y}_1(t, \tau, \sigma) + \epsilon^2 \mathbf{y}_2(t, \tau, \sigma) + O(\epsilon^3),$$

and the basic goal is to calculate a leading order approximation $\mathbf{y}_0(t, \epsilon t, \epsilon^2 t)$ which is formally valid over time scales $t \sim O(\epsilon^{-2})$.

The $\mathrm{ord}(1)$ and $\mathrm{ord}(\epsilon)$ equations in asymptotic hierarchy proceed in a straightforward way [3], after which $\mathbf{y}_0$ is determined up to function $\mathbf{A}(\sigma)$ and $\mathbf{y}_1$ is determined up to function $\mathbf{B}(\tau, \sigma)$. At $\mathrm{ord}(\epsilon^2)$, one starts with

$$\frac{\partial \mathbf{y}_2}{\partial t} = \mathsf{L}\mathbf{y}_2 + \mathbf{y}_1 \cdot \nabla \mathbf{f}(\mathbf{y}_0, t, 0) + \frac{\partial \mathbf{f}}{\partial \epsilon}(\mathbf{y}_0, t, 0) - \frac{\partial \mathbf{y}_1}{\partial \tau} - \frac{\partial \mathbf{y}_0}{\partial \sigma}.$$

We proceed by simply systematically imposing the nonsecularity conditions that $\epsilon^2 \mathbf{y}_2 \ll \epsilon \mathbf{y}_1$ and $\epsilon \mathbf{y}_1 \ll \mathbf{y}_0$ as $\epsilon \downarrow 0$ over the desired domain of validity $t \sim O(\epsilon^{-2})$, $\tau \sim O(\epsilon^{-1})$, $\sigma \sim O(1)$, and enforcing appropriate initial conditions. We introduce no auxiliary *ad hoc* assumptions such as the zeroing of homogenous solutions (as is criticized in [3]). After imposing non-secularity for $\mathbf{y}_2$ with respect to t, we obtain a linear, variable coefficient equation for $\mathbf{B}(\tau, \sigma)$ with respect to τ.

Applying now the non-secularity condition on the τ variable dependence of $\mathbf{B}$ for $\tau \sim O(\epsilon^{-1})$ can encounter two possible difficulties: it may be impossible to remove secularity (due to exploding homogenous solutions) [1], or the resulting condition does not completely determine $\mathbf{A}(\sigma)$. The latter case occurs in the Duffing oscillator and an oscillator with weak cubic damping. However, if one proceeds to the $\mathrm{ord}(\epsilon^3)$ equation, one does obtain a complete determination of $\mathbf{y}_0$ over time scales $t \sim O(\epsilon^{-2})$ which moreover is found to be accurate when compared with an asymptotic expansion of the exact solution in these test problems. By contrast, the extension of the method of averaging to three time scales developed in [4] actually gives an incorrect second order correction to the renormalized frequency in the Duffing oscillator.

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2 Some Details for Duffing Equation

We develop a multiple-scale analysis through three time scales on the equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x + \epsilon x^3 = 0, \qquad (1)$$

$$x(t=0) = x_{\mathrm{in}}, \; \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = v_{\mathrm{in}}$$

We first rewrite the equation in terms of $z = x + \mathrm{i}\dot{x}$ [3]

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\mathrm{i}z - \frac{1}{8}\mathrm{i}\epsilon(z+\bar{z})^3,$$

$$z(t=0) = x_{\mathrm{in}} + \mathrm{i}v_{\mathrm{in}}$$

After processing the $\mathrm{ord}(1)$ and $\mathrm{ord}(\epsilon)$ equations, we have

$$\begin{aligned} z_0(t,\tau,\sigma) &= A(\tau,\sigma)\mathrm{e}^{-\mathrm{i}t}, & A(\tau,\sigma) &= \rho_A(\sigma)\mathrm{e}^{-\frac{3}{8}\mathrm{i}(\rho_A(\sigma))^2\tau - \mathrm{i}\psi_A(\sigma)}, \\ \rho_A(\sigma=0) &= \sqrt{x_{\mathrm{in}}^2 + v_{\mathrm{in}}^2}, & \psi_A(\sigma=0) &= \tan^{-1}(v_{\mathrm{in}}/x_{\mathrm{in}}), \\ z_1(t,\tau,\sigma) &= B(\tau,\sigma)\mathrm{e}^{-\mathrm{i}t} + z_{1,\mathrm{p}}(A(\tau,\sigma),\bar{A}(\tau,\sigma),t), & z_1(t=0,\tau=0,\sigma=0) &= 0, \end{aligned}$$

Solving the $\mathrm{ord}(\epsilon^2)$ equation, we obtain:

$$z_2 = \left[\frac{51}{256}iA^3\bar{A}^2 - \frac{3}{8}iA^2\bar{B} - \frac{3}{4}i|A|^2B - \frac{\partial A}{\partial\sigma} - \frac{\partial B}{\partial\tau}\right]e^{-it}t + z_{2,\mathrm{NS}},$$

where $z_{2,\mathrm{NS}}$ consists of terms bounded with respect to t. Nonsecularity with respect to $t \lesssim \mathrm{ord}(\epsilon^{-2})$ implies:

$$\frac{\partial B}{\partial\tau} + \frac{\partial A}{\partial\sigma} = \frac{51}{256}iA^3\bar{A}^2 - \frac{3}{8}iA^2\bar{B} - \frac{3}{4}iA\bar{A}B$$

Solving for the dependence of B on τ [3], we obtain:

$$\frac{B}{A} = \left(\mathrm{i}\frac{\partial\phi_A}{\partial\sigma} + \frac{51}{256}\mathrm{i}\rho_A^4 - \frac{3}{8}\mathrm{i}H(\sigma) - \frac{1}{2}\frac{1}{\rho_A^2}\frac{\partial\rho_A^2}{\partial\sigma}\right)\tau + \frac{3}{16}i\frac{\partial\rho_A^2}{\partial\sigma}\tau^2 + \frac{\mathrm{i}}{2\rho_A^2}J + \frac{1}{2\rho_A^2}H$$

for real-valued functions $H(\sigma)$ and $J(\sigma)$ to be determined. Imposing nonsecularity with respect to $\tau \lesssim \mathrm{ord}(\epsilon^{-1})$ yields:

$$A(\tau,\sigma) = \sqrt{x_{\mathrm{in}}^2 + v_{\mathrm{in}}^2}\,\mathrm{e}^{-\mathrm{i}\phi_A(\tau,\sigma)}, \qquad \phi_A(\tau,\sigma) = \frac{3}{8}\rho_A^2\tau - \frac{51}{256}\rho_A^4\sigma + \int_0^\sigma \frac{3}{8}H(\sigma')\mathrm{d}\sigma' + \chi$$

where $H(\sigma)$ and constant χ remain to be determined. By naive equation counting, we would have expected by now to determine A completely, but the phase function ϕ_A is not yet well determined. If we proceed similiarly to $\mathrm{ord}(\epsilon^3)$ and simply impose simple nonsecularity conditions as above, we then obtain $H = 5/16$ and $\chi = \tan^{-1}(v_{\mathrm{in}}/x_{\mathrm{in}})$, completing the determination of A and thereby the leading order approximation of the solution to Eq. (1) over time scales $t \sim O(\epsilon^{-2})$. For the present problem, this amounts to computing the renormalized frequency (correctly) through $O(\epsilon^2)$.

3 Conclusions

We demonstrate that the method of multiple scales can be pursued to obtain a leading order approximation valid over three time scales in some weakly nonlinear test problems, but the asymptotic analysis must be pursued to one higher order than one would expect by counting the numbers of equations and unknowns. Of course the test problems can be solved more efficiently by methods such as Poincare-Lindstedt, but our purpose is to examine an asymptotic method which can generalize to partial differential equations. Indeed, this work was motivated by a desire to understand some peculiarities which emerged when we attempted the method of multiple scales with three time scales to some partial differential equations in turbulent diffusion.

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