HOMOGENIZATION THEORY FOR A REPLENISHING PASSIVE SCALAR FIELD

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Abstract. Homogenization theory provides a rigorous framework for calculating the effective diffusivity of a decaying passive scalar field in a turbulent or complex flow. We extend this framework to the case where the passive scalar fluctuations are continuously replenished by a source (and/or sink). The basic structure of the homogenized equations carries over, but in some cases the homogenized source can involve a non-trivial coupling of the velocity field and the source. We derive expressions for the homogenized source term for various multiscale source structures and interpret them physically.

1. Introduction

Enhanced diffusion of scalar fields in turbulent and chaotic flows is a problem of fundamental importance in plasma physics, engineering, astrophysics, and the geosciences, and developing techniques to model this phenomenon represents a significant theoretical and practical challenge [1, 2]. The simplest framework, which underlies the general problem, is one in which the scalar field is advected passively, without disturbing the flow pattern, so that the scalar density (or scalar field) $\theta(\boldsymbol{x},t)$ satisfies the passive advection-diffusion equation.

$$\partial_t\theta + \boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\theta = \mathrm{Pe}^{-1}\Delta\theta, \quad \theta(\boldsymbol{x},t=0) = \theta_{\mathrm{in}}(\boldsymbol{x}), \tag{1}$$

We have nondimensionalized this equation in terms of the magnitude V of the velocity field and a characteristic length scale L_v on which the velocity field varies, so that the nondimensionalized diffusion coefficient is expressed in terms of the Peclét number $\mathrm{Pe} = L_v V/\kappa$, where κ is the dimensional diffusivity, and the initial scalar density is given by $\theta_{\mathrm{in}}(\boldsymbol{x})$. Suitable boundary conditions are also prescribed (such as boundedness on an infinite spatial domain). The flow $\boldsymbol{v}(\boldsymbol{x},t)$ is assumed incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$) and is prescribed by a kinematic model. Equation (1) can describe the dispersal of pollutants in the atmosphere and ocean, chemical species in stellar interiors, or dyes used to visualize flow patterns in a laboratory, to name a few examples. Moreover, passive scalar advection is an important test-bed for deepening our understanding of mixing and transport phenomena in complex flows.

Typically, one seeks to obtain a description of the long-time, large-scale evolution of $\theta(\boldsymbol{x},t)$ in terms of the statistics of the velocity field. This task is simplified considerably when the length scale of variation of the velocity field (1 in nondimensionalized units) is strongly separated from that of the passive scalar density (δ^{-1}). In that case, we represent the passive scalar field in terms of spatial variables adapted to this scale, a corresponding diffusive time scale and amplitude rescaled according to the number d of spatial dimensions:

$$\theta^{(\delta)}(\boldsymbol{x},t) \equiv \delta^{-d}\theta(\boldsymbol{x}/\delta, t/\delta^2). \tag{2}$$

This rescaled passive scalar density satsifes the rescaled advection-diffusion equation:

$$\partial_t \theta^{(\delta)} + \delta^{-1}\boldsymbol{v}\left(\frac{\boldsymbol{x}}{\delta}, \frac{t}{\delta^2}\right) \cdot \boldsymbol{\nabla}\, \theta^{(\delta)} = \mathrm{Pe}^{-1}\Delta\theta^{(\delta)}.$$

One can now employ multiscale methods to separate the dynamics on fast ($\boldsymbol{\xi} = \delta^{-1}\boldsymbol{x}, \tau = \delta^{-2}t$) and slow $(\boldsymbol{x}, t)$ space and time scales, where $\delta \ll 1$ acts as a scale-separation parameter, through a multiple-scale expansion

$$\theta^{(\delta)}(\boldsymbol{x},t) = \left[\overset{(0)}{\theta}(\boldsymbol{\xi},\tau;\boldsymbol{x},t) + \delta\overset{(1)}{\theta}(\boldsymbol{\xi},\tau;\boldsymbol{x},t) + \delta^2\overset{(2)}{\theta}(\boldsymbol{\xi},\tau;\boldsymbol{x},t)\cdots\right]\Bigg|_{\boldsymbol{\xi}=\delta^{-1}\boldsymbol{x},\tau=\delta^{-2}t} \tag{3}$$

Following this prescription, one develops a *homogenization theory* which shows that the solution to Eq. (1) is well approximated (for sufficently large scale separation δ^{-1}) over at time scales $t \lesssim O(1)$ by the solution $\bar{\theta}(\boldsymbol{x},t)$ of the following homogenized equation, in which the effects of the advective term in (1) are replaced by an effective diffusivity tensor K^* [2, 3, 4, 5, 6]:

$$\partial_t\bar{\theta} = \mathrm{Pe}^{-1}\,\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left(\mathsf{K}^*\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\bar{\theta}\right), \tag{4}$$

The effective diffusivity tensor can be represented as follows:

$$\mathsf{K}^*_{ij} = \delta_{ij} + \bar{\mathsf{K}}_{ij} \tag{5a}$$

where the first term represents an isotropic component due to molecular diffusion and the second term an *enhanced diffusivity* given by the non-negative definite, symmetric tensor

$$\bar{\mathsf{K}}_{ij} = \langle \boldsymbol{\nabla}_{\boldsymbol{\xi}}\chi_i \cdot \boldsymbol{\nabla}_{\boldsymbol{\xi}}\,\chi_j\rangle, \tag{5b}$$

where angle brackets denote an average over the fast space and time variables $(\boldsymbol{\xi}, t)$ and the vector $\boldsymbol{\chi}(\boldsymbol{\xi},\tau)$ is the unique, mean-zero solution to the auxiliary *cell problem*

$$\left(\partial_\tau + \boldsymbol{v}(\xi,\tau)\cdot\boldsymbol{\nabla}_{\boldsymbol{\xi}} - \mathrm{Pe}^{-1}\Delta_{\boldsymbol{\xi}}\right)\boldsymbol{\chi}(\boldsymbol{\xi},\tau) = -\boldsymbol{v}(\boldsymbol{\xi},\tau). \tag{5c}$$

with suitable boundary conditions. For the case in which the velocity $\boldsymbol{v}$ is periodic in both space and time, $\boldsymbol{\chi}$ is also taken to be periodic, and the average $\langle\cdot\rangle$ in Eq. (5b) is an average over the spatiotemporal period of the velocity field. A similar framework applies also for random velocity fields with finite correlation length [3, 7, 8, 9, 10, 11] but we will consider here the periodic context for the sake of minimizing technical concerns.

In this study, we extend homogenization theory to address the problem of passive scalar advection in the presence of a source-sink distribution in the bulk (which we will generally call simply "source" for simplicity, with negative value corresponding to sinks),

$$\partial_t\theta + \boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}\theta = \mathrm{Pe}^{-1}\Delta\theta + \sigma s(\boldsymbol{x},t). \tag{6}$$

Here, $\sigma = SL_v/(V\Theta)$, with S the magnitude of the source strength and Θ the magnitude of the passive scalar field fluctuations, is a non-dimensional parameter quantifying the strength of the prescribed source term $s(\boldsymbol{x},t)$ relative to the other terms in the equation.

It may be thought of as the passive scalar equivalent of a Damköhler number in chemically reactive flows in that it expresses the ratio of the flow advection time L_v/V to the time scale Θ/S on which the source replenishes the passive scalar field.

Examples of replenishing scalar fields include: weak temperature fluctuations emanating from a distributed heat source [12], sea-surface temperature relaxing to atmospheric equilibrium [13]. and plankton blooms associated with upwelling nutrients in the ocean [14]. As the passive scalar field in (6) is continuously replenished by $s(\boldsymbol{x}, t)$, it can achieve a statistically stationary equilibrium between production and dissipation [15]. Consequently, if the source itself is solely (or predominantly) responsible for the passive scalar fluctuations, the magnitude of the response Θ is actually naturally linked to the magnitude of the source S, so σ is better thought of as an endogenously determined nondimensional parameter rather than an exogenously prescribed governing parameter. For the more general case where other influences (such as spatial boundary conditions or initial conditions) predominantly set the magnitude of the passive scalar fluctuations, our main development still applies if we decompose the passive scalar evolution (through linearity) into its responses from initial and boundary conditions and to the source.

The key question we shall address is how the source and small-scale velocity field interact in determining the homogenized equation for the passive scalar field on large scales. The motivation for this pursuit arose from the apparent difference in effective diffusion (as measured by mixing efficiencies) in passive scalar dynamics with a source [15] as compared to without a source [2, 8, 16] . As it turns out, these discrepancies can be reconciled through a careful consideration of relevant physical time scales for the replenishing and mixing dynamics [17], but this still leaves open the question of how the homogenized equation appears in the presence of a source. We will see that in fact the effective diffusivity persists unchanged from the source-free case, but that the homogenized source term can exhibit interesting coupling between the small-scale velocity and source fluctuations in a manner reminiscent of skew-flux terms [18, 19, 20, 21, 22, 23]. In [24], we address the apparent discrepancies between the behavior of homogenized diffusivity as developed in [2, 8, 16] and the mixing efficiency measures in [15] from the perspective of these homogenized equations by exploring the role of scale separation and anisotropy in the flow.

The various homogenized passive scalar equations that can emerge under a source/sink term will be reviewed together with physical interpretations in Section 2. The derivation of these results from a multiple scales analysis is developed in Section 3. Some concluding remarks are offered in Section 4.

2. Homogenized Passive Scalar Equations with Source

We will allow for the passive scalar source term to depend both on the spatial scale of the velocity as well as a larger spatial scale, separated by a factor δ. Similarly, we allow the passive scalar source to vary in time both on the same time scale as the velocity field as well as a slower time scale separated by a factor η:

$$s \equiv \delta^d s^{(\delta)}(\boldsymbol{x}, t; \delta\boldsymbol{x}, \eta t). \tag{7}$$

The amplitude scaling has been chosen so that the source function $s^{(\delta)}$ expresses the rate of passive scalar density production on the large length scale δ^{-1}, noting how the passive scalar density amplitude is rescaled (2) according to the spatial dimension. For simplicity, we will generally assume that the velocity $\boldsymbol{v}$ and the source $s^{(\delta)}$ are periodic with respect to the small-scale variables, though the case of random variations can also be handled through homogenization theory provided the fields are statistically homogenous and stationary with finite correlation length. The results change only in that "periodic" solutions are replaced by "statistically homogenous and stationary" solutions, and small-scale averages become statistical averages rather than averages over a period cell.

After performing the diffusive rescaling (Eq. (2)) to the large scale, we obtain the following advection-diffusion equation with source:

$$\partial_t \theta^{(\delta)} + \delta^{-1} \boldsymbol{v}\left(\frac{\boldsymbol{x}}{\delta}, \frac{t}{\delta^2}\right) \cdot \boldsymbol{\nabla}\, \theta^{(\delta)} = \mathrm{Pe}^{-1} \Delta \theta^{(\delta)} + \sigma \delta^{-2} s^{(\delta)}\left(\frac{\boldsymbol{x}}{\delta}, \frac{t}{\delta^2}; \boldsymbol{x}, \frac{\eta t}{\delta^2}\right) \tag{8}$$

The goal of homogenization theory is here to obtain an approximate equation for $\theta^{(\delta)}$ that depends only on the slow variables and averages out the fast dependencies. We will first discuss in Subsection 2.1 the results for the case in which the slow time scale of the source coincides with the diffusive time scale associated to the spatial scale of the source: $\eta = \delta^2$. Although this conveniently reduces the number of time scales in Eq. 8 to two: t/δ^2 and t, it is not apparently favored by any physical reason. Therefore, we subsequently discuss the cases $\eta \ll \delta^2$ and $\eta \gg \delta^2$ in Subsections 2.2 and 2.3, respectively. These turn out to be rather straightforward modifications to the results developed for the special case $\eta = \delta^2$, which may therefore be considered as a mathematically distinguished limit for the asymptotic problem (8) with $\delta \downarrow 0$. In all cases, we seek homogenized approximations that are valid at least for bounded intervals of time as measured in terms of the time variable characterizing the large scales $t \sim O(1)$, which is just the diffusion time scale associated with the large spatial scale. Derivations for the results that follow will be presented in Section 3.

2.1. Distinguished Limit of Coalescing Time Scales: $\eta = \delta^2$.

For the case in which the time scale of the fluctuations of the source coincides with the diffusion time scale associated with its large spatial scale, the homogenized form of Eq. (8) takes the general form:

$$\partial_t \bar{\theta} = \mathrm{Pe}^{-1} \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \left(\mathsf{K}^* \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \bar{\theta}\right) + \hat{\sigma} S^*(\boldsymbol{x}, t). \tag{9}$$

The effective diffusivity tensor K^* is identical to that derived for the freely decaying problem, that is, it has the form given by (5a,5b) involving the solution to the cell problem (5c). The new term $\hat{\sigma} S^*$ appearing in (9) represents a homogenized source and depends on the properties of the source field. In presenting the possible cases, we recall that the nondimensional parameter σ, involving the strength of source relative to response, is an endogenous parameter.

2.1.1. *Source with Nontrivial Small-Scale Mean.* When the source does not vanish upon averaging over the space and time scales of the velocity field, then the homogenized source is equal to this simple average:

$$S^*(\boldsymbol{x},t) \equiv \langle s^{(\delta)}(\boldsymbol{\xi},\tau;\boldsymbol{x},t)\rangle \tag{10}$$

and the nondimensionalized source strength $\sigma = \hat{\sigma}\delta^2$, with order unity constant $\hat{\sigma}$, for self-consistency. This means the time scale for replenishing the passive scalar field is δ^2 slower than the advection time, and is therefore comparable with the diffusion time scale over the large spatial scale. This match between time scales is consistent with the homogenized source being a simple coarse-grained average.

2.1.2. *Source with Zero Small-Scale Mean.* If the average of the source over the space and time scales of the velocity field vanishes, then the direct interaction of the source with the passive scalar field gives trivial results on the large scales. Rather, the passive scalar response is relatively weaker ($\sigma = \hat{\sigma}\delta$ with order unity constant $\hat{\sigma}$) and described by the homogenized equation (9) with effective source term

$$S^* = -\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \langle \boldsymbol{v}\theta_{\sigma 0}\rangle \tag{11}$$

where $\theta_{\sigma 0} = \theta_{\sigma 0}(\boldsymbol{\xi},\tau;\boldsymbol{x},t)$ is the unique, mean-zero, periodic solution to the cell problem

$$\left(\partial_\tau + \boldsymbol{v}(\boldsymbol{\xi},\tau)\cdot\boldsymbol{\nabla}_{\boldsymbol{\xi}} - \mathrm{Pe}^{-1}\Delta_{\boldsymbol{\xi}}\right)\theta_{\sigma 0} = s^{(\delta)}(\boldsymbol{\xi},\tau;\boldsymbol{x},t). \tag{12}$$

Note this is an equation only with respect to the small-scale variables $\boldsymbol{\xi}$ and τ; the dependence on the large-scale variables $\boldsymbol{x}$ and t is purely parametric. Consequently, unlike Eq. (8), the computational solution of Eq. (12) is not a stiff multiscale problem, but rather a partial differential equation posed entirely on the small scale. The fact that $\sigma \sim \mathrm{ord}(\delta)$ implies that the time scale on which the passive scalar field is replenished is δ^{-1} in our rescaled units, intermediate between the time scales characterizing the small and large scales. This helps explain why the homogenized source is related in a bit more complex way to the original source than it was for the previous case in Subsubsection 2.1.1.

Note that $\theta_{\sigma 0}$ is exactly the local response of the passive scalar field to the source on the small scale, with the large-scale variation simply frozen at its local value. Consequently, $\boldsymbol{v}\theta_{\sigma 0}$ is precisely the advective flux of passive scalar density generated in response to the local behavior of the source. Were the source to have no large-scale spatial variation ($s = s(\boldsymbol{\xi},\tau;t)$) then the passive scalar response $\theta_{\sigma 0}$ would likewise be independent of the large scale spatial variable $\boldsymbol{x}$, so the advective flux $\langle \boldsymbol{v}\theta_{\sigma 0}\rangle$ generated by the source would likewise be independent of $\boldsymbol{x}$, rendering the expression (11) equal to zero. But large-scale variations in the source will induce large-scale variations in the advective flux $\boldsymbol{v}\theta_{\sigma 0}$ generated by the leading order small-scale passive scalar response, yielding a generically nonzero effective source (11) on the large scales. This effective source is seen to be exactly minus the large-scale divergence of an advective flux, which, when averaged over small scales, provides a rectified contribution in spite of the fact that both the velocity field and the passive scalar response (as well as the source that generates it!) have zero mean when averaged over the small scales. That the effective source should be the negative divergence of the passive scalar flux follows from general principles from continuum mechanics [25] whereby fluxes only generate a variation in the field when the net flux into a small control volume does not balance the net flux out. The diffusive flux generated by the leading order small-scale passive scalar response is equal to $-\mathrm{Pe}^{-1}\boldsymbol{\nabla}_{\boldsymbol{\xi}}\theta_{\sigma 0}$, which always vanishes when averaged over the small scales, and thereby makes no contribution to the large-scale evolution of the passive scalar density.

The effective source (11) takes a particularly interesting form when the source takes the special form of a small-scale variation $\alpha(\boldsymbol{\xi},\tau)$ with zero mean $\langle\alpha\rangle = 0$, modulated by a large scale envelope $A(\boldsymbol{x},t)$:

$$\tilde{s}(\boldsymbol{\xi};\boldsymbol{x}) = A(\boldsymbol{x},t)\alpha(\boldsymbol{\xi},\tau), \tag{13}$$

Then the solution to the cell problem (12) can be expressed as

$$\theta_{\sigma 0} = A(\boldsymbol{x},t)\beta(\boldsymbol{\xi},\tau), \tag{14a}$$

where $\beta(\boldsymbol{\xi},\tau)$ is the mean-zero, periodic solution to the small-scale equation

$$\left(\partial_\tau + \boldsymbol{v}(\boldsymbol{\xi},\tau)\cdot\boldsymbol{\nabla}_{\boldsymbol{\xi}} - \mathrm{Pe}^{-1}\Delta_{\boldsymbol{\xi}}\right)\beta(\boldsymbol{\xi},\tau) = \alpha(\boldsymbol{\xi},\tau). \tag{14b}$$

The homogenized source is then simply

$$S^*(\boldsymbol{x}) = -\boldsymbol{\nabla}_{\boldsymbol{x}}\left(\boldsymbol{F}A(\boldsymbol{x},t)\right) = -\boldsymbol{F}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}A(\boldsymbol{x},t), \tag{15a}$$

with

$$\boldsymbol{F} = \langle\boldsymbol{v}\beta\rangle. \tag{15b}$$

The structure of the homogenized source (Eqs. (11) or (15)) is somewhat reminiscent of skew-flux terms [18, 19, 20, 21, 22, 23] in the homogenized advection-diffusion equation, in that it represents the divergence of an upscaled flux which would vanish in the absence of large-scale modulation or boundaries. By contrast, the usual symmetric component of the effective diffusivity (5b) obtained from upscaled passive scalar fluxes is generally non-vanishing in the absence of large-scale variations in the environment. Just as effective drift generated by the the skew-flux is proportional to the gradient of large-scale modulations, so is the homogenized source proportional in strength to the gradient of the large-scale modulation of the source.

2.1.3. *Special Subcase of Source with Nonzero Small-Scale Mean.* A somewhat peculiar case arises for the case of a nonzero small-scale mean when the homogenized source term (11) vanishes identically. Three situations in which this may occur are as follows:

- the source has no large scale variation $s^{(\delta)} = s^{(\delta)}(\boldsymbol{\xi},t)$,
- the average advective flux of the leading order passive small-scale scalar response $\langle\boldsymbol{v}\theta_{\sigma 0}\rangle$ vanishes identically, perhaps by a symmetry of the velocity field on small scales
- the average advective flux is directed perpendicularly to the large-scale variation of the source; in the case of a large-scale modulated source of the form (13), this can be expressed as $\boldsymbol{F}\perp\boldsymbol{\nabla}_{\boldsymbol{x}}A(\boldsymbol{x},t)$.

In this case, the nondimensionalized strength σ of the source in Eq. (8) must by self-consistency be an order unity constant $\sigma = \hat{\sigma}$, and the rescaled passive scalar field is represented to leading order by the sum of a purely large-scale variation $\bar{\theta}(\boldsymbol{x}, t)$ as well as a term $\theta_{\sigma 0}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)$ with a mean zero small-scale variation:

$$\theta^{(\delta)}(\boldsymbol{x}, t) = \bar{\theta}(\boldsymbol{x}, t) + \sigma\theta_{\sigma 0}\left(\delta^{-1}\boldsymbol{x}, \delta^{-2}t; \boldsymbol{x}, t\right) + O(\delta) \tag{16}$$

for $t \sim O(1)$. The term $\theta_{\sigma 0}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)$ is the solution to Eq. (17b), while $\bar{\theta}$ satisfies the homogenized equation (9) with effective source given by

$$S^*(\boldsymbol{x}) = -\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \langle \boldsymbol{v}\theta_{\sigma 1}\rangle, \tag{17a}$$

with $\theta_{\sigma 1}$ the unique, mean-zero, periodic solution to the equation

$$\left(\partial_\tau + \boldsymbol{v}(\boldsymbol{\xi}, \tau) \cdot \boldsymbol{\nabla}_{\boldsymbol{\xi}} - \mathrm{Pe}^{-1}\Delta_{\boldsymbol{\xi}}\right)\theta_{\sigma 1} = \left(2\mathrm{Pe}^{-1}\boldsymbol{\nabla}_{\boldsymbol{\xi}} - \boldsymbol{v}(\boldsymbol{\xi}, \tau)\right) \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}\theta_{\sigma 0}. \tag{17b}$$

The term $\theta_{\sigma 0}$, as discussed in Subsubsection 2.1.2, simply describes the small-scale response of the passive scalar field to the source, and here it contributes to leading order. The purely large-scale response $\bar{\theta}$ is driven by a homogenized source (17a) which now involves the divergence of the average advective flux generated by the first order correction $\theta_{\sigma 1}$ to the small-scale passive scalar response. This can be understood simply from the fact that the divergence of the average advective flux generated by the leading order small-scale passive scalar response vanishes by assumption of the subcase under consideration. Note that if the passive scalar field is observed on large scales (coarse-graining over the small scales), then only $\bar{\theta}$ will contribute to the leading order passive scalar field fluctuations. This large-scale variation will be driven by a nontrivial effective source (17a) again only if the source itself has nontrivial large-scale variations, but unlike the case discussed in Subsubsection 2.1.2, the effective source (17a) will be generically nonvanishing when this is the case. This can be seen most simply for the case where the original source is expressed in modulated form $s(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) = A(\boldsymbol{x}, t)\alpha(\boldsymbol{\xi}, \tau)$ so that $\theta_{\sigma 0}$ can be expressed as in Eq. (14) and we can write

$$\theta_{\sigma 1} = -\boldsymbol{\gamma}(\boldsymbol{\xi}, \tau) \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} A(\boldsymbol{x}, t), \tag{18}$$

where $\boldsymbol{\gamma}$ satisfies,

$$\left(\partial_\tau + \boldsymbol{v}(\boldsymbol{\xi}, \tau) \cdot \boldsymbol{\nabla}_{\boldsymbol{\xi}} - \mathrm{Pe}^{-1}\Delta_{\boldsymbol{\xi}}\right)\boldsymbol{\gamma} = \left(-2\mathrm{Pe}^{-1}\boldsymbol{\nabla}_{\boldsymbol{\xi}} + \boldsymbol{v}\right)\beta. \tag{19}$$

The homogenized source then becomes

$$S^*(\boldsymbol{x}, t) = \mathsf{G} : \boldsymbol{\nabla}\boldsymbol{\nabla} A(\boldsymbol{x}, t) \tag{20}$$

with symmetric tensor

$$\mathsf{G} = \frac{1}{2}\langle \boldsymbol{v} \otimes \boldsymbol{\gamma} + \boldsymbol{\gamma} \otimes \boldsymbol{v}\rangle. \tag{21}$$

This tensor can be checked to be nonvanishing even under the presence of reflection or (for the random case) rotational small-scale symmetries of the velocity field and source which would cause the homogenized source (11) to vanish.

For this case, the time scale ratio σ between the advection time and time of replenishing the passive scalar field is order unity, meaning that the passive scalar is responding substantially to the source on the same time scale as that characterizing the small scales, which explains why the leading order passive scalar density here in fact depends on the fast space and time scales as well as the slow ones.

We see for the case in which the source has purely small-scale, mean zero variations $s = s(\boldsymbol{\xi}, \tau)$, the leading order passive scalar field response $\theta_{\sigma 0}$ is also purely small-scale with mean zero. In this case, the large-scale response of the passive scalar field coarse-grained over the small scales is actually transcendentally small in δ, making no appearance anywhere in the asymptotic expansion. This can be understood by noting that if the passive scalar response is concentrated on the small-scale δ^{-1} and is smooth, then the magnitude of its fluctuations at the widely separated scale of order unity is transcendentally small because its Fourier transform will decay faster than algebraically away from its dominant scale.

2.2. **Temporal fluctuations in source slower than diffusive $\eta \ll \delta^2$.** If the source fluctuates on a time scale which is slower than the diffusion time scale associated with the large spatial scale of the source, then all the above equations remain valid over the order unity time scale in rescaled coordinates, except that the large-scale component of the source behaves quasistatically. That is, over the diffusive time scales described above, the source is to be treated as frozen with respect to its large-scale time variable, though with nontrivial dependence on the slow variable τ.

2.3. **Temporal fluctuations in source faster than diffusive $\delta^2 \ll \eta \ll 1$.** When the slow-scale temporal fluctuations in the source are faster than the diffusive time scale associated to the large spatial scale, then the leading order passive scalar field still obeys the effective equations described above except that the averages defining the effective source are now averaged not only over the small space and time scales but also over the slower time scale of source fluctuation as well. The effective source therefore only depends on the large-scale spatial variable and is independent of time: $S^* = S^*(\boldsymbol{x})$. Equivalently, the behavior of the passive scalar density is the same as if the original source were replaced by a source averaged over its slow-scale time fluctuation (but not over the faster space and time fluctuations which can interact nontrivially with the velocity field fluctuations on these same scales).

3. Homogenization calculation for a replenishing passive scalar field

The derivation of the homogenized advection-diffusion equation in the absence of a source is described in detail in [2]. In this section, we extend this formulation to treat the case of a replenishing passive scalar field.

We seek a solution to (8) in the limit of wide separation $\delta \to 0$ between the larger spatial scale of the source and the spatial scale of the velocity field. To that end, we introduce again the multiscale expansion (3) of the passive scalar density in terms of the fast variables

$(\boldsymbol{\xi}, \tau)$ and the slow variables $(\boldsymbol{x}, t)$:

$$\theta^{(\delta)}(\boldsymbol{x}, t) = \theta^{(\delta)}_{\mathrm{MS}}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)\Big|_{\boldsymbol{\xi}=\delta^{-1}\boldsymbol{x}, \tau=\delta^{-2}t}$$

$$\theta^{(\delta)}(\boldsymbol{x}, t) = \left[\overset{(0)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) + \delta \overset{(1)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) + \delta^2 \overset{(2)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) \cdots\right]\Big|_{\boldsymbol{\xi}=\delta^{-1}\boldsymbol{x}, \tau=\delta^{-2}t}$$

The space and time derivatives acting on the multiscale expansion coefficients $\overset{(j)}{\theta}$ transform as follows under the chain rule:

$$\nabla\theta^{(\delta)}(\boldsymbol{x}, t) = \left(\nabla_{\boldsymbol{x}} + \delta^{-1}\nabla_{\boldsymbol{\xi}}\right)\theta^{(\delta)}_{\mathrm{MS}}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)\Big|_{\boldsymbol{\xi}=\delta^{-1}\boldsymbol{x}, \tau=\delta^{-2}t}, \tag{22}$$

$$\partial_t\theta^{(\delta)}(\boldsymbol{x}, t) = \left(\partial_t + \delta^{-2}\partial_\tau\right)\theta^{(\delta)}_{\mathrm{MS}}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)\Big|_{\boldsymbol{\xi}=\delta^{-1}\boldsymbol{x}, \tau=\delta^{-2}t}. \tag{23}$$

The advection-diffusion equation thereby transforms into the multiscale form

$$\left(\mathcal{Q}_0 + \delta\mathcal{Q}_1 + \delta^2\mathcal{Q}_2\right)\theta^{(\delta)}_{\mathrm{MS}}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) = \sigma s(\boldsymbol{\xi}, \tau; \boldsymbol{x}, \eta\delta^{-2}t). \tag{24}$$

where

$$\mathcal{Q}_0 = \partial_\tau + \boldsymbol{v}(\xi, \tau)\cdot\nabla_{\boldsymbol{\xi}} - \mathrm{Pe}^{-1}\Delta_{\boldsymbol{\xi}}, \tag{25}$$

$$\mathcal{Q}_1 = \boldsymbol{v}(\xi, \tau)\cdot\nabla_{\boldsymbol{x}} - 2\mathrm{Pe}^{-1}\nabla_{\boldsymbol{\xi}}\cdot\nabla_{\boldsymbol{x}}, \tag{26}$$

$$\mathcal{Q}_2 = \partial_t - \mathrm{Pe}^{-1}\Delta_{\boldsymbol{x}}. \tag{27}$$

The multiscale analysis is now organized depending on how the ratio σ of the advection time scale to the replenishing time of the source, and the ratio η of the time scales in the source fluctuations, compare to the spatial scale separation parameter δ. As discussed in Section 2, while $\eta \ll 1$ may be prescribed arbitrarily, the parameter σ must be determined by self-consistency. We proceed to describe the multiscale derivations in parallel with the results presented in Section 2.

For simplicity, the derivations will assume the velocity and source is periodic with respect to the small-scale variables (with common space and time periods), but the arguments extend directly to at least formally establish analogous results for the case for the case of statistically homogenous and stationary random small-scale fluctuations with finite correlation length.

3.1. Distinguished Limit of Coalescing Time Scales: $\eta = \delta^2$.

3.1.1. *Source with Nontrivial Small-Scale Mean.* Here $\sigma = \hat{\sigma}\delta^2$, where $\hat{\sigma} \sim \mathrm{ord}(1)$, provides a self-consistent solution (in which the rescaled passive scalar density $\theta^{(\delta)}$ has nontrivial leading order behavior). Equating terms in (24) order by order yields:

$$\mathrm{ord}(1): \quad \mathcal{Q}_0 \overset{(0)}{\theta} = 0, \tag{28}$$

$$\mathrm{ord}(\delta): \quad \mathcal{Q}_0 \overset{(1)}{\theta} = -\mathcal{Q}_1 \overset{(0)}{\theta}, \tag{29}$$

$$\mathrm{ord}(\delta^2): \quad \mathcal{Q}_0 \overset{(2)}{\theta} = -\mathcal{Q}_2 \overset{(0)}{\theta} - \mathcal{Q}_1 \overset{(1)}{\theta} + \hat{\sigma}s^{(\delta)}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t). \tag{30}$$

We will neglect transients (as they decay on the fast time scale) and thereby assume the small-scale fluctuations in the passive scalar field expansion coefficients are purely periodic, as driven by the periodic small-scale fluctuations in the velocity field and source. In solving the asymptotic hierarchy, we will invoke the following principle:

Lemma 1 (Solvability Condition)**.** *Given a function g with the same periodicity with respect to the small space and time variables $\boldsymbol{\xi}$ and τ as $\boldsymbol{v}$, the equation*

$$\mathcal{Q}_0 f = g \tag{31}$$

has a solution $f(\boldsymbol{\xi},\tau)$ with the same periodicity if and only if g has zero mean, meaning that the average of g over a space-time period cell vanishes. Moreover, this solution is unique up to an additive constant.

The basic idea behind the lemma is that on a suitable function space of periodic functions of $\boldsymbol{\xi}$ and τ, the operator $\mathcal{Q}_0$ is essentially rank-one deficient, in that it and its adjoint have a one-dimensional null space consisting of constants (with respect to the small-scale variables $\boldsymbol{\xi}$ and τ). Proving this for the case of a steady velocity field can be done by standard Sobolev space techniques for the associated elliptic operator [26], while the proof for the time-dependent case is actually rather subtle because the periodic boundary conditions in τ are a bit unusual for a parabolic operator [27].

Proceeding now to invoke this solvability condition on the ord(1) equation (28), we conclude that $\overset{(0)}{\theta}$ must be a constant with respect to the small-scales, meaning it depends only on the large scales (which enter as parameters as far as the solution of equations of the form (31) are concerned):

$$\overset{(0)}{\theta}(\boldsymbol{\xi},\tau;\boldsymbol{x},t) \equiv \bar{\theta}(\boldsymbol{x},t)\,. \tag{32}$$

Equation (29) then becomes

$$\mathcal{Q}_0 \overset{(1)}{\theta}(\boldsymbol{\xi},\tau;\boldsymbol{x},t) = -\boldsymbol{v}(\boldsymbol{\xi},\tau)\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\,\bar{\theta}(\boldsymbol{x},t)\,. \tag{33}$$

As noted above, the appearance of the $\boldsymbol{x}$ and t variables are purely parametric as far as the operator $\mathcal{Q}_0$ is concerned. Therefore, noting that $\boldsymbol{v}$ is a mean-zero periodic function, we can define $\boldsymbol{\chi}(\boldsymbol{\xi},\tau)$ as the unique, periodic, mean-zero solution of the auxiliary *cell problem*

$$\mathcal{Q}_0\boldsymbol{\chi}(\boldsymbol{\xi},\tau) = -\boldsymbol{v}(\boldsymbol{\xi},\tau). \tag{34}$$

and establish the solution to Eq. (29) as

$$\overset{(1)}{\theta}(\boldsymbol{\xi},\tau;\boldsymbol{x},t) = \bar{\theta}_1(\boldsymbol{x},t) + \boldsymbol{\chi}(\boldsymbol{\xi},\tau)\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\,\bar{\theta}(\boldsymbol{x},t)\,, \tag{35}$$

Thus, the ord(δ) correction to the passive scalar field θ depends on the fast scales only through the vector $\boldsymbol{\chi}$, the solution to the cell problem (34).

Solvability of the ord(δ^2) equation (30) requires that the right-hand side vanishes when averaged over fast scales:

$$\langle \mathcal{Q}_2 \overset{(0)}{\theta} + \mathcal{Q}_1 \overset{(1)}{\theta} - \hat{\sigma} s^{(\delta)}(\boldsymbol{\xi},\tau;\boldsymbol{x},t)\rangle = 0. \tag{36}$$

Substituting the solutions (32) and (35) into (36) and using the fact that the velocity field has mean zero then gives the following equation for the zeroth-order approximation of the passive scalar field

$$\partial_t \bar{\theta} = \mathrm{Pe}^{-1}\, \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{\cdot} \left(\mathsf{K}^* \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{x}}\, \bar{\theta}\right) + \hat{\sigma} S^*(\boldsymbol{x}, t), \tag{37}$$

where $S^* = \langle s^{(\delta)}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)\rangle$ and the effective diffusivity K^* is determined in exactly the same way (5) as for the freely evolving passive scalar field without source. (One simply needs to note that K^* is being expressed in the standard way as the identity matrix plus the symmetric part of the tensor $-\langle \boldsymbol{v} \otimes \boldsymbol{\chi}\rangle$, which can be written as the expression in Eq. (5) after substituting in for $\boldsymbol{v}$ using the cell problem (34) and integration by parts [2]).

3.1.2. *Source with Zero Small-Scale Mean.* If the source has the property of averaging to zero over the small-scales ($\langle s(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)\rangle = 0$), then the source does not consistently maintain the passive scalar density at an order unity level under the previous scaling $\sigma = \hat{\sigma}\delta^2$, so a stronger relative source strength is required. We will find $\sigma = \hat{\sigma}\delta$, with $\hat{\sigma} \sim \mathrm{ord}(1)$, to be typically sufficient.

The asymptotic hierarchy generated by Eq. (24) now reads:

$$\mathrm{ord}(1): \quad \mathcal{Q}_0 \overset{(0)}{\theta} = 0, \tag{38}$$

$$\mathrm{ord}(\delta): \quad \mathcal{Q}_0 \overset{(1)}{\theta} = -\mathcal{Q}_1 \overset{(0)}{\theta} + \hat{\sigma} s^{(\delta)}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t), \tag{39}$$

$$\mathrm{ord}(\delta^2): \quad \mathcal{Q}_0 \overset{(2)}{\theta} = -\mathcal{Q}_2 \overset{(0)}{\theta} - \mathcal{Q}_1 \overset{(1)}{\theta}. \tag{40}$$

The ord(1) equation is the same as in Subsection 3.1.1, and we again conclude that, to lowest order in δ, the passive scalar field is a function of the slow scales only:

$$\overset{(0)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) \equiv \bar{\theta}(\boldsymbol{x}, t). \tag{41}$$

The ord(δ) equation is now

$$\mathcal{Q}_0 \overset{(1)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) = -\boldsymbol{v}(\boldsymbol{\xi}, \tau) \boldsymbol{\cdot} \boldsymbol{\nabla}_{\boldsymbol{x}}\, \bar{\theta} + \hat{\sigma} s(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t), \tag{42}$$

which differs from Eq. (29) only in the presence of the source term on the right hand side. By linearity of the equation, we may express the full solution as:

$$\overset{(1)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) = \bar{\theta}_1(\boldsymbol{x}, t) + \boldsymbol{\chi}(\boldsymbol{\xi}, \tau) \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \bar{\theta}(\boldsymbol{x}, t) + \hat{\sigma}\theta_{\sigma 0}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t), \tag{43}$$

where $\boldsymbol{\chi}$ again is the unique mean-zero periodic solution to the cell problem (34) and for each value of $(\boldsymbol{x}, t)$, $\theta_\sigma(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t)$ is the unique, periodic, mean-zero solution to another cell problem:

$$\mathcal{Q}_0 \theta_{\sigma 0} = s(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t). \tag{44}$$

This equation has such a unique solution according to the solvability condition because for the case under consideration, the source has zero small-scale mean.

Proceeding to $\mathrm{ord}(\delta^2)$, the solvability condition gives

$$\langle \mathcal{Q}_2 \overset{(0)}{\theta} + \mathcal{Q}_1 \overset{(1)}{\theta} \rangle = 0, \tag{45}$$

which, upon computing the averages using the expressions (41) and (42) for $\overset{(0)}{\theta}$ and $\overset{(1)}{\theta}$, yield the homogenized equation:

$$\partial_t \bar{\theta} = \mathrm{Pe}^{-1} \, \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \left(\mathsf{K}^* \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \, \bar{\theta} \right) + \hat{\sigma} S^*(\boldsymbol{x}), \tag{46}$$

where K^* is the same effective diffusivity tensor as before but now the homogenized source reads

$$S^*(\boldsymbol{x}) = - \, \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \langle \boldsymbol{v} \theta_{\sigma 0} \rangle \tag{47}$$

where $\theta_{\sigma 0}$ is the solution of the second cell problem (44).

3.1.3. *Special Subcase of Source with Nonzero Small-Scale Mean.* When $\langle s^{(\delta)}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) \rangle = 0$ and

$$- \, \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \langle \boldsymbol{v} \theta_{\sigma} \rangle$$

vanishes, then again the source is not maintaining the passive scalar density at an order unity level with the nondimensional strength $\sigma = \delta \hat{\sigma}$. We consider next whether taking the nondimensional source strength $\sigma = \hat{\sigma}$ of order unity will self-consistently generate a passive scalar density with order unity magnitude.

Once again, equating terms of the same order in δ in Eq. (24) under the new scaling of σ gives

$$\mathrm{ord}(1): \quad \mathcal{Q}_0 \overset{(0)}{\theta} = \hat{\sigma} s(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t), \tag{48}$$

$$\mathrm{ord}(\delta): \quad \mathcal{Q}_0 \overset{(1)}{\theta} = -\mathcal{Q}_1 \overset{(0)}{\theta}, \tag{49}$$

$$\mathrm{ord}(\delta^2): \quad \mathcal{Q}_0 \overset{(2)}{\theta} = -\mathcal{Q}_2 \overset{(0)}{\theta} - \mathcal{Q}_1 \overset{(1)}{\theta}. \tag{50}$$

so that the source term now enters at ord(1). Immediately we see that the ord(1) result no longer holds: the leading order behavior of the passive scalar field is now a function of both the fast and slow scales.

We can write the solution to Eq. (48) as

$$\overset{(0)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) = \bar{\theta}(\boldsymbol{x}, t) + \hat{\sigma} \theta_{\sigma 0}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t), \tag{51}$$

where $\theta_{\sigma 0}$ is the unique mean-zero, periodic solution to Eq. (44).

At $\mathrm{ord}(\delta)$ we now have

$$\mathcal{Q}_0 \overset{(1)}{\theta} = -\boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \bar{\theta} - \hat{\sigma} \mathcal{Q}_1 \theta_{\sigma 0}. \tag{52}$$

Again, we write

$$\overset{(1)}{\theta}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t) = \bar{\theta}_1(\boldsymbol{x}, t) + \boldsymbol{\chi}(\boldsymbol{\xi}, \tau) \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \bar{\theta}(\boldsymbol{x}, t) + \hat{\sigma} \theta_{\sigma 1}(\boldsymbol{\xi}, \tau; \boldsymbol{x}, t), \tag{53}$$

where $\boldsymbol{\chi}$ is the unique mean-zero periodic solution to the same cell problem (29) as before, and now

$$\mathcal{Q}_0\theta_{\sigma 1} = -\mathcal{Q}_1\theta_{\sigma 0}. \tag{54}$$

Equation (54) will itself have a solution only if the small-scale average of the right hand side, which can be shown to be $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\langle \boldsymbol{v}\theta_{\sigma 0}\rangle$, vanishes. But this is exactly the condition that corresponds to the special case under consideration. We may therefore take $\theta_{\sigma 1}$ to be the unique mean-zero, periodic solution of Eq. (54).

The solvability equation at $\mathrm{ord}(\delta^2)$ now gives

$$\partial_t\bar{\theta} = \mathrm{Pe}^{-1}\,\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left(\mathsf{K}^*\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\,\bar{\theta}\right) + \hat{\sigma}S^*(\boldsymbol{x},t), \tag{55}$$

where, once more, the effective diffusivity tensor K^* takes the same form as before. Now, however,

$$S^*(\boldsymbol{x},t) = -\,\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\langle \boldsymbol{v}\theta_{\sigma 1}\rangle \tag{56}$$

represents the large-scale production of $\bar{\theta}$. Note that this effective term is determined by the *coupled hierarchy* of cell problems:

$$\mathcal{Q}_0\theta_{\sigma 1} = -\mathcal{Q}_1\theta_{\sigma 0}, \tag{57}$$
$$\mathcal{Q}_0\theta_{\sigma 0} = s^{(\delta)}(\boldsymbol{\xi},\tau;\boldsymbol{x},t). \tag{58}$$

3.2. **Temporal fluctuations in source slower than diffusive $\eta \ll \delta^2$.** The derivations are essentially unchanged other than noting that the source should be treated as parameterically frozen with respect to its slow time argument since under the diffusive rescaling (2), it is evaluated at $\eta t/\delta^2$which does not vary substantially over any of the $t \lesssim \mathrm{ord}(1)$ time scales under consideration.

3.3. **Temporal fluctuations in source faster than diffusive $\delta^2 \ll \eta \ll 1$.** Formally this case requires the introduction of an intermediate time scale $\tilde{t} = \eta\delta^{-2}t$ to resolve the slower temporal fluctuations in the source:

$$\theta^{(\delta)}(\boldsymbol{x},t) = \theta^{(\delta)}_{\mathrm{MS}}(\boldsymbol{\xi},\tau;\boldsymbol{x},\tilde{t},t)\Big|_{\boldsymbol{\xi}=\delta^{-1}\boldsymbol{x},\tilde{t}=\eta\delta^{-2}t,\tau=\delta^{-2}t}$$

with the corresponding introduction of three time derivatives in the multiscale analysis:

$$\partial_t\theta^{(\delta)}(\boldsymbol{x},t) = \left(\partial_t + \eta\delta^{-2}\partial_{\tilde{t}} + \delta^{-2}\partial_\tau\right)\theta^{(\delta)}_{\mathrm{MS}}(\boldsymbol{\xi},\tau;\boldsymbol{x},t)\Big|_{\boldsymbol{\xi}=\delta^{-1}\boldsymbol{x},\tilde{t}=\eta\delta^{-2}t,\tau=\delta^{-2}t}.$$

This complicates the development of the asymptotic hierarchy a bit, but one can check in every case that in fact the leading order passive scalar density does not depend on the intermediate time variable $\tilde{t}$. Because the slow time scale of variation of the source is still much faster than the diffusive time scale on which the passive scalar density has nontrivial variations, the small-scale averages defining the various homogenized source terms must also in the present case average over the slow-time scale of variation in the source. One reason this case behaves so simply is that the time scale of variation of the source $\delta^2\eta^{-1}$ does not resonate with any other time scale in the system, so generates no effect on larger time scales and can be averaged away without loss.

4. Conclusions

We have extended the homogenization of a passive scalar field to include the presence of a source distributed throughout the bulk which replenishes the passive scalar field, possibly in a multiscale manner in space and time. We find here that for a source (or sink) varying on a length scale large compared to the velocity field advecting the passive scalar field, homogenized equations can be developed which describe the leading order passive scalar density. In these homogenized equations, the effective diffusivity is exactly the same as in the freely evolving passive scalar case. When the source has nontrivial mean when averaged over the small scale of the velocity field, then the homogenized equation takes the obvious form (Subsubsection 2.1.1) in which the source is simply replaced by its average over small scales. When, however, the source has mean zero over the small scales characterizing the velocity field (as is the case in many studies such as [15] and the Kraichnan model (see the references in Section 4 of [2]), the source and velocity field fluctuations can couple to produce a nontrivial large-scale effective source ((15) or (17) that in some ways is reminiscent of skew-flux corrections to effective drift [18, 19, 20, 21, 22, 23]). Nonetheless, if the source varies purely on the scale of the velocity field (and has no large-scale modulation), the effective source terms developed here all become trivial. The homogenization theory is essentially looking at large scales and misses the bulk of the passive scalar fluctuations which are occuring on smaller spatial scales.

This gives one reason why the mixing efficiency bounds developed in [15] are not in conflict with homogenization theory – those bounds were applied to the case in which the velocity and source fluctuations are on the same scale, and homogenization theory does not appear appropriate for quantifying the passive scalar response in this situation. [17] offers a careful study of how the mixing efficiency measures behave under various ratios of scale separation between the velocity and the source, and sufficiently strong scale separation ($\delta \ll \mathrm{Pe}^{-1}$) is shown to produce agreement with homogenization theory, consistent with the results developed here. Indeed, if this scale separation condition is violated (so that $\delta \gtrsim \mathrm{Pe}^{-1}$), then the formal asymptotic expansion above breaks down because the terms proportional to Pe^{-1} would have to be moved to different levels of the asymptotic hierarchy. Further perspectives concerning homogenization theory for a passive scalar field replenished by a source/sink field, with a view toward physical interpretation, will be developed in [24]

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