NL3284 **Fokker–Planck equation**

The Fokker–Planck equation (FPE) plays a role in stochastic systems analogous to that of the Liouville equation in deterministic mechanical systems. Namely, the FPE describes in a statistical sense how a *collection* of initial data evolves in time. To be precise, let the phase space describing the system be parameterized by the state vector $\boldsymbol{y}$, and let $\boldsymbol{Y}(t)$ denote the value of the state vector assumed by the system at time t. Suppose the dynamics of the system is governed by a stochastic differential equation (SDE) of the form:

$$\mathrm{d}\boldsymbol{Y}(t) = \boldsymbol{U}(\boldsymbol{Y}(t),t)\,\mathrm{d}t + \sigma(\boldsymbol{Y}(t),t)\cdot \mathrm{d}\boldsymbol{W}(t). \tag{1}$$

The deterministic component of the dynamics is described by the drift vector $\boldsymbol{U}(\boldsymbol{y},t)$, while the random part is driven by a vector of independent Brownian motions $\boldsymbol{W}(t)$ which are coupled to the system through the matrix $\sigma(\boldsymbol{y},t)$.

Alternatively, we can define the system dynamics as a “diffusion process” in the relevant phase space such that:

$$\begin{cases} \lim_{\tau\downarrow 0}\tau^{-1}\langle \boldsymbol{Y}(t+\tau)-\boldsymbol{Y}(t)\rangle & = \boldsymbol{U}(\boldsymbol{Y}(t),t), \\ \lim_{\tau\downarrow 0}\tau^{-1}\langle (\boldsymbol{Y}(t+\tau)-\boldsymbol{Y}(t))\otimes(\boldsymbol{Y}(t+\tau)-\boldsymbol{Y}(t))\rangle & = 2\mathsf{D}(\boldsymbol{Y}(t),t), \\ \lim_{\tau\downarrow 0}\tau^{-1}\langle |\boldsymbol{Y}(t+\tau)-\boldsymbol{Y}(t)|^{\gamma}\rangle & = 0 \text{ for } \gamma > 2, \end{cases} \tag{2}$$

where $\langle\cdot\rangle$ denotes an average over the random component of the dynamics. The SDE and diffusion process descriptions are equivalent, with $2\mathsf{D} = \sigma\sigma^{\dagger}$.

One useful way to describe the evolution of stochastic systems over a finite (rather than infinitesmal) time interval is through the probability transition density $p(t,\boldsymbol{y}|t',\boldsymbol{y}')$, which describes the likelihood that if the state variable assumes the value $\boldsymbol{y}'$ at time t', then it will assume a value near $\boldsymbol{y}$ at the later time t. More precisely, p is defined in terms of conditional probability as:

$$\text{Prob}\,(\boldsymbol{Y}(t)\in B|\boldsymbol{Y}(t')=\boldsymbol{y}') = \int_B p(t,\boldsymbol{y}|t',\boldsymbol{y}')\,\mathrm{d}\boldsymbol{y} \tag{3}$$

for all nice (Borel) sets B in the phase space. The FPE, also known as the Kolmogorov forward equation, is a partial differential equation which describes how the probability transition density evolves when the system can be described as an SDE (1) or diffusion process (2):

$$\begin{cases} \frac{\partial p(t,\boldsymbol{y}|t',\boldsymbol{y}')}{\partial t} & = \frac{\partial}{\partial \boldsymbol{y}}\left(-\boldsymbol{U}(\boldsymbol{y},t)p + \frac{\partial}{\partial \boldsymbol{y}}\left(\mathsf{D}(\boldsymbol{y},t)p\right)\right), \\ p(t=t',\boldsymbol{y}|t',\boldsymbol{y}') & = \delta(\boldsymbol{y}-\boldsymbol{y}'). \end{cases} \tag{4}$$

This equation is to be solved for $t > t'$ with fixed data for the source variables $(t',\boldsymbol{y}')$ and appropriate boundary conditions in $\boldsymbol{y}$ (Risken, 1989, Ch. 4).

The probability transition density can be used to describe the probability density for the state vector at any moment of time:

$$\text{Prob}\,(\boldsymbol{Y}(t)\in B) = \int_B \phi(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{5}$$

by simply integrating against the prescribed probability distribution of the states at the initial time t':

$$\phi(t, \boldsymbol{y}) = \int p(t, \boldsymbol{y}|t', \boldsymbol{y}')\phi(t', \boldsymbol{y}')\,\mathrm{d}\boldsymbol{y}'. \tag{6}$$

Alternatively, $\phi(t, \boldsymbol{y})$ can be shown to satisfy the FPE (4) but with initial data $\phi(t', \boldsymbol{y}')$ prescribed more generally as a nonnegative function with integral one rather than as a delta function.

The FPE was first applied to describe Brownian motion. In the most idealized case, where inertia is completely neglected, Einstein showed that the statistics of the position $\boldsymbol{x} = \boldsymbol{X}(t)$ of a Brownian particle obeys a FPE which coincides with the ordinary diffusion equation:

$$\frac{\partial \phi(t, \boldsymbol{x}|t', \boldsymbol{x}')}{\partial t} = \kappa \Delta_{\boldsymbol{x}} \phi \tag{7}$$

where $\Delta_{\boldsymbol{x}}$ is the Laplace operator and the diffusion coefficient $\mathsf{D} = \kappa \mathsf{I}$ is a constant scalar multiple of the identity matrix. The stochastic differential description of this model is simply:

$$\mathrm{d}\boldsymbol{X}(t) = (2\kappa)^{1/2}\,\mathrm{d}\boldsymbol{W}(t). \tag{8}$$

The effects of inertia and external forces can be incorporated by passing to a phase space description including both the position $\boldsymbol{x} = \boldsymbol{X}(t)$ and the velocity $\boldsymbol{v} = \boldsymbol{V}(t)$ of the Brownian particle. The equations of motion can then be written in terms of Newton's law with a random forcing component proportional to $\mathrm{d}\boldsymbol{W}$:

$$\begin{cases} \mathrm{d}\boldsymbol{X}(t) & = \boldsymbol{V}(t)\,\mathrm{d}t, \\ m\,\mathrm{d}\boldsymbol{V}(t) & = -m\xi \boldsymbol{V}(t)\,\mathrm{d}t + \boldsymbol{f}(\boldsymbol{X}(t), t)\,\mathrm{d}t + (2k_{\mathrm{B}}Tm\xi)^{1/2}\,\mathrm{d}\boldsymbol{W}(t), \end{cases} \tag{9}$$

where ξ is a friction coefficient, k_{B} is Boltzmann's constant, T is absolute temperature, and $\boldsymbol{f}(\boldsymbol{x}, t)$ represents the deterministic part of the external applied force. The equivalent Fokker–Planck description for the phase space probability transition density in this system reads:

$$\frac{\partial p(t, \boldsymbol{x}, \boldsymbol{v}|t', \boldsymbol{x}', \boldsymbol{v}')}{\partial t} = \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot \left[\left(\xi \boldsymbol{v} - \frac{\boldsymbol{f}(\boldsymbol{x}, t)}{m} + \frac{k_{\mathrm{B}}T\xi}{m}\boldsymbol{\nabla}_{\boldsymbol{v}}\right) p\right] - \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot (\boldsymbol{v}p) \tag{10}$$

Observe how this equation generalizes the Liouville equation for deterministic mechanics to include Brownian motion. The simplified diffusion equation (7) can be obtained through an asymptotic limit of the full FPE (10) with $\kappa = k_{\mathrm{B}}T/(m\xi)$, when $\boldsymbol{f}(\boldsymbol{x}, t) = 0$ and $(k_{\mathrm{B}}T/(m\xi^2\ell^2))^{1/2} \ll 1$ with ℓ a characteristic length of the system (Bocquet, 1997; Risken, 1989). A similar reduction to a partial differential equation in coordinate space, called the Smoluchowski equation, is also possible for collections of interacting particles where the friction tensor depends on the particle configuration (Titulaer, 1980). From the FPE for Brownian motion, one can compute various statistical properties such as its mean-square displacement, the spectrum of its fluctuations, rate of relaxation of initial velocity to thermal equilibrium values, and the probability distribution for the time at which the particle first achieves a certain location or surmounts a potential barrier (Wax, 1954).

The FPE has found useful application in computing similar statistical properties in numerous other systems such as lasers, polymers, particle suspensions, quantum electronic systems, molecular motors, and finance. In some instances, the system is not at first formulated in one of the senses (1) or (2) which immediately imply a FPE. Rather, the systems are more naturally represented in terms of a master equation with a complete description of the rates at which the system (randomly) jumps from one state to another. The FPE is an appropriate approximation to this system when certain asymptotic conditions, such as small jumps, a large system size, or a separation of time scales between resolved and unresolved variables, is met (Grabert, 1982; Risken, 1989; van Kampen, 1981).

A variety of techniques for practical analysis of FPEs can be found in Risken (1989). The main alternative to analyzing stochastic systems through the FPE is through the stochastic differential (or Langevin) formulation (1). The FPE has the merit of being a deterministic partial differential equation, which can be solved once through either analytical or numerical methods. The stochastic differential description (1) by contrast consists of ordinary differential equations for which individual realizations (samples) are faster to simulate through Monte Carlo methods (Kloeden & Platen, 1992). But obtaining a statistical description of the general behavior of the system requires the generation of a large number of realizations.

Certain statistical quantities described above can be easily calculated using either the solution of the FPE or by operating directly on the FPE itself. On the other hand, the stochastic differential system (1) must be used if one wishes to characterize particular realizations of the stochastic system. Moreover, the stochastic differential description can be readily generalized to cases in which the random driving has temporal correlations (Kubo *et al.*, 1991; Moss & McClintock, 1989). The FPE formulation can be modified to a "fractional" formalism for certain self-similar temporal correlation structures (Metzler & Klafter, 2000), but further generalizations are much more complicated and usually require some sort of approximation (Moss & McClintock, 1989).

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See also Brownian motion; Nonequilibrium statistical mechanics; Phase space diffusion and correlations; Stochastic processes

Further Reading

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Kloeden, P.E. & Platen, E. 1992. *Numerical Solution of Stochastic Differential Equations*, Berlin and New York: Springer [sections 1.7, 2.4, 4.1, 4.7]

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Reduction to Smoluchowski equation for multiple particles

van Kampen, N.G. 1981. *Stochastic Processes in Physics and Chemistry*, Amsterdam: North-Holland [chap. 8]

Careful discussion of physical applicability of FPE

Wax, N. (editor). 1954. *Selected Papers on Noise and Stochastic Processes*, New York: Dover

Collection of early papers applying stochastic techniques to statistical mechanics