NL3281 **Brownian motion**

Robert Brown, a leading botanist, observed in 1828 that a wide variety of particles suspended in liquid exhibit an intrinsic, irregular motion when viewed under a microscope. While not the first to witness such motion, his experimental focus on this phenomenon which would bear his name established its universality and intrinsic nature, thereby raising it as an issue for fundamental scientific inquiry (Nelson, 1967). Deutsch has recently raised the question of whether Brown really witnessed Brownian motion or fluctuations due to some external contaminating influence (Peterson, 1991). Indeed, while Brownian motion is an ubiquitous phenomenon, not all irregular motions can be ascribed to Brownian motion. The dancing of dust particles in sunlight is dominated by imperceptible turbulent currents, not Brownian motion. True Brownian motion is generally only visible on scales of microns and below, but has important macroscopic ramifications because all microscopic particles manifest it. For example, Brownian motion makes possible both the fine-scale mixing of initially segregated substances in nature and industry, as well as the passive transport of ions, nutrients, and fuel which allow biological cells to support life.

The origin of Brownian motion remained under debate throughout the 19th century, with Cantoni, Delsaux, Gouy, and C. Weiner proposing that thermal motions in the suspending liquid were responsible, as discussed in Einstein (1956, pp. 86–88), Gallavotti (1999, ch. 8), and Russel *et al.* (1989, pp. 65–66). Attempts to examine this hypothesis quantitatively were hampered by the inability to measure accurately the velocity of particles undergoing Brownian motion, since such motion loses coherence over time scales (microseconds) which are shorter than those which experimental observations were able to resolve. In one of five ground-breaking papers that Einstein published in 1905, he offered a statistical mechanics means for theoretical calculations involving Brownian motion (Einstein, 1956). Einstein realized that the quantity involving Brownian motion which can be best observed under a microscope in an experiment is the "diffusivity":

$$D = \lim_{t\to\infty} \frac{|X(t) - X(0)|^2}{2t} \tag{1}$$

where $X(t)$ denotes the observed displacement of the Brownian particle along a fixed direction at time t. In practice, t is simply taken as some satisfactorily long time of observation, and there is no need for fine temporal resolution as there would be if the velocity were to be measured. Einstein employed a random walk model for his analysis and showed that the diffusivity defined in (1) is identical to the diffusion constant that describes the macroscopic evolution of the concentration density $n(\boldsymbol{x}, t)$ of a large number of Brownian particles:

$$\frac{\partial n(\boldsymbol{x}, t)}{\partial t} = D \sum_{j=1}^{3} \frac{\partial^2 n(\boldsymbol{x}, t)}{\partial x_j^2}. \tag{2}$$

Through an elegant argument based on *equilibrium* statistical mechanical arguments, Einstein showed that the diffusivity D of a Brownian particle must be related to its friction coefficient ξ in the following way:

$$D = \frac{k_{\mathrm{B}}T}{m\xi} \tag{3}$$

where k_{B} is Boltzmann's constant and T is the absolute temperature (measured in Kelvin scale), and m is the particle's mass. The friction coefficient ξ appears in the relation between the drag force F_{drag} and velocity v of the particle in steady state motion (assuming a low Reynolds number):

$$F_{\mathrm{drag}} = m\xi v. \tag{4}$$

For a sphere of radius a moving through a fluid with dynamic viscosity μ, the friction coefficient is given by $\xi = 6\pi\mu a/m$. The remarkable property of the "Einstein relation" in (3) is that it links a quantity D pertaining to statistically unpredictable dynamical fluctuations to a quantity ξ which involves deterministic, steady-state properties. Later work generalized the Einstein relation (3) to "fluctuation–dissipation theorems" which express the structure of the spontaneous statistical fluctuations in a wide class of physical systems to the structure of the dissipative (frictional) dynamics (Kubo *et al.*, 1991, ch. 1).

The basic theory of Brownian motion was developed by Einstein in 1905, a time when the premises of the atomic theory of matter were still not yet fully agreed upon (Gallavotti, 1999; Nelson, 1967). Einstein realized that a careful observation of Brownian motion and his relation between the diffusivity of a Brownian particle and its mobility could be used to calculate the number of particles making up a given mass of fluid if the atomic theory were valid. Under a microscope sufficient to resolve the Brownian motion of a particle, all quantities in (3) are directly observable except for Boltzmann's constant k_{B}. Therefore, the Einstein relation (3) can be used to compute a value for k_{B} based on Brownian motion data. Now, k_{B} is in turn related to Avogadro's number N_A, which is the number of molecules in a "mole" (a certain well-defined macroscopic amount) of a substance. The Brownian motion data and the Einstein relation therefore furnish an independent prediction for Avogadro's number N_A and thereby the number of molecules per unit mass of the fluid. In other words, the number (and therefore mass) of the individual fluid particles could be calculated without having to observe them at an individual level, an experimental feat which has become possible only in recent years. Instead, their individual mass and number could be assessed through their collective influence on a much larger and therefore observable immersed particle. Jean Perrin experimentally confirmed in 1908 that the value of N_A computed in this way agreed with those obtained from other techniques (Gallavotti, 1999), providing strong support for the atomic theory of matter. Since the 1970s, Brownian motion has been investigated in the laboratory through dynamic light scattering techniques (Russel *et al.*, 1989, ch. 3).

Figure 1. Sample trajectories of fractional Brownian motion using Fourier-wavelet method (Elliott *et al.*, 1997). Top panel: $H = \frac{1}{3}$, lower panel:$H = \frac{2}{3}$. Both simulations used the same random numbers.

The most idealized mathematical representation of Brownian motion with diffusivity D is defined as $(2D)^{1/2}W(t)$, where $W(t)$ is a canonical continuous random process with Gaussian statistics such that $W(0) = 0$, $\langle W(t) \rangle = 0$, and

$$\langle (W(t) - W(t'))^2 \rangle = |t - t'|. \tag{5}$$

This mathematical Brownian motion is often referred to as the Wiener process (Borodin & Salminen, 2002; Nelson, 1967; Gallavotti, 1999). This idealized Brownian motion has independent increments (no inertia). Physical Brownian motion of course has some small inertia as well as several other complicating influences from the fluid environment and presence of other nearby Brownian particles (Russel *et al.*, 1989). These extra features can be built into a dynamical description using the mathematical Brownian motion as the basic noise input with influence mediated by the other physical parameters. The mathematical Brownian motion serves a similar role in modeling noise input in a wide variety of stochastic models in physics, biology, finance, and other fields. More precisely, the Levy–Khinchine theorem indicates that in any system which is affected by noise in a continuous way such that the noise on disjoint time intervals is independent can be modeled in terms of mathematical Brownian motion (Reichl, 1998, ch. S4, 5).

Discontinuous noise-induced jumps, in contrast, are modeled in terms of Poisson processes or more generally Levy processes (Reichl, 1998, ch. S4, 5). Continuous noise with long-range correlations (so that the independent increment property is not satisfied), on the other hand, can often be usefully modeled in terms of "fractional Brownian motion" (FBM) (Mandelbrot, 2002). This is an idealized Gaussian random process $Z(t)$ with $Z(0) = 0$, $\langle Z(t) \rangle = 0$, and

$$\langle (Z(t) - Z(t'))^2 \rangle = |t - t'|^{2H} \tag{6}$$

where the Hurst exponent H is chosen from the interval $0 < H < 1$. The FBM with $H = \frac{1}{2}$ corresponds to ordinary Brownian motion with independent increments. FBM's with $\frac{1}{2} < H < 1$ have positive, long-ranged correlations with less rough trajectories and large excursions, while FBM's with $0 < H < \frac{1}{2}$ have negative, long-ranged correlations with rougher trajectories and a more oscillatory character (Figure 1). All FBM's have a statistical self-similarity property; the statistics of the rescaled FBM $\lambda^{-H} Z(\lambda t)$ are identical to those of the original $Z(t)$. That is, these processes have no finite length or time scale associated to them, and can be thought of as random fractals. Fractional Brownian motions are therefore particularly appropriate for modeling systems with fluctuations occurring over a wide range of scales; cutoff lengths can be introduced by filtering an input FBM. Models built from FBM's have been developed in turbulence theory, natural landscape and cloud structures, surface adsorption processes, neural signals in biology, and self-organized critical systems such as earthquakes, forest fires, and sandpiles.

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See also Fluctuation-dissipation theorem; Fluid dynamics; Fokker–Planck equation; Random walks

Further Reading

Borodin, A.N. & Salminen, P. 2002. *Handbook of Brownian motion: Facts and Formulae*, 2nd edn., Basel: Birkhäuser

Einstein, A. 1956. *Investigations on the Theory of the Brownian Movement*, edited with notes by R. Fürth, translated by A.D. Cowper, New York: Dover

Elliott, Jr, F.W., Horntrop, D.J. & Majda, A.J. 1997. A Fourier-wavelet Monte Carlo method for fractal random fields. *Journal of Computational Physics*, 132(2): 384–408

Gallavotti, G. 1999. *Statistical Mechanics*, Berlin: Springer

Kubo, R., Toda, M. & Hashitsume, N. 1991. *Statistical Physics*, 2 vols, 2nd edn., Berlin: Springer

Mandelbrot, B.B. 2002. *Gaussian Self-affinity and Fractals*, chap. IV. New York: Springer

Nelson, E. 1967. *Dynamical Theories of Brownian Motion*, Princeton, N.J.: Princeton University Press

Peterson, I. 1991. Did Brown see Brownian motion? *Science News*, 139: 287

Reichl, L.E. 1998. *A Modern Course in Statistical Physics*. 2nd edn., New York: Wiley

Russel, W.B., Saville, D.A. & Schowalter, W.R. 1989. *Colloidal Dispersions*. Cambridge and New York: Cambridge University Press