Parametrization for Mesoscale Ocean Transport through Random Flow Models

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We describe a mathematical approach based on homogenization theory toward representing the effects of mesoscale coherent structures on large-scale transport in the ocean. We demonstrate the approach on a deterministic and a random model flow.

1 Introduction

We are interested in parameterizing the transport effects of sub-grid scale flow structures appearing in mesoscale oceanic turbulence. We follow a bottom-up approach, starting with simple flow models, each representing some aspect of turbulent flow, and aim to gradually increase the complexity of our flow models.

Turbulent diffusion of passive scalar fields such as heat, pollution, salinity is governed by the advection diffusion equation[2]

$$\partial_t T(\vec{x},t) + \vec{U}(\vec{x},t)\cdot\vec{\nabla}T(\vec{x},t) = \kappa\Delta T(\vec{x},t) \tag{1}$$

where $T(\vec{x},t)$ denotes the passive scalar field, $\vec{U}(\vec{x},t)$ the velocity field of the flow medium, and κ a "molecular" diffusion coefficient which we will take here to also include turbulent transport by structures with length scales below hundreds of meters. Ideally, $\vec{U}(\vec{x},t)$ is the solution of the Navier-Stokes equation and involves a broad range of scales.

Our motivation comes from the fact that current ocean circulation models cannot resolve scales below about 100 km in the ocean whereas much of ocean's turbulent energy lies on scales around or less than this scale, inducing fluxes that affect all aspects of ocean circulation and hence the associated climate models. To have an idea about the underlying statistics of the climate one inevitably faces the need to parameterize the influence of mesoscale turbulence. While homogenization theory is not applicable to systems such as atmospheric or Kolmogorov turbulence with power-law distribution of energy with respect to length scale, it is a plausible approach to capturing transport effects by coherent structures with characteristic length scales in mesoscale oceanic turbulence.

In Section 2, we present two flow models. In the deterministic Rankine Vortex Model, we are able to parameterize effective diffusivity explicitly as a function of a single parameter which is a combination of underlying flow parameters. In the random Poisson Blob Model, we are also able to parameterize effective diffusivity as a function of a single parameter which is a combination of underlying flow parameters in a regime of modest Peclét number. In section 3, we present our findings for these two flow models.

2 Setting

Rankine Vortex Model: Consider a steady state velocity field $\vec{U}(\vec{x}) = \vec{V} + \vec{v}(\vec{x})$ consisting of a sum of a constant large scale mean drift $\vec{V}$ along the $(1,1)$-direction, and small scale deterministic, periodic local fluctuations $\vec{v}(\vec{x})$ given by a compactly supported Rankine Vortex $\vec{v}(\vec{x}) = \nabla^{\perp}\psi(\vec{x})$ with nondimensionalized stream function taken as a periodized version of $\psi(x,y) = 12[\frac{(x^2+y^2)^{3/2}}{9} - \frac{x^2+y^2}{6}]$.

The non-dimensional model flow parameters are $a = v_{\text{vor}}/V$, relative strength of local fluctuations to mean flow and the Peclét number, $\text{Pe} = (L\cdot V)/\kappa$, where v_{vor} and V are magnitudes of the small-scale and mean velocity, respectively, and L is the period length. In this case, the non-dimensional advection-diffusion equation describing the evolution of passive scalar field with large scale initial condition is given by

$$\partial_t T(\vec{x},t) + (\vec{V} + a\cdot\vec{v}(\vec{x}))\cdot\vec{\nabla}T(\vec{x},t) = \text{Pe}^{-1}\Delta T(\vec{x},t), \quad T(\vec{x},0) = T_{in}(\vec{x}). \tag{2}$$

Poisson Blob Model [1, 3]**:** For our simplest random flow model, we take a straightforward generalization of the above single vortex field. By randomizing the locations of the single vortices and superposing the resulting velocity fields, we construct a steady, homogenous random velocity field $\vec{v}(\vec{x}) = \nabla^{\perp}\psi$ to model random local small-scale fluctuations. The nondimensionalized stream function $\psi(x,y) = \sum_1^N \psi_p(x+x_p, y+y_p)$ is a sum of N Rankine vortices,

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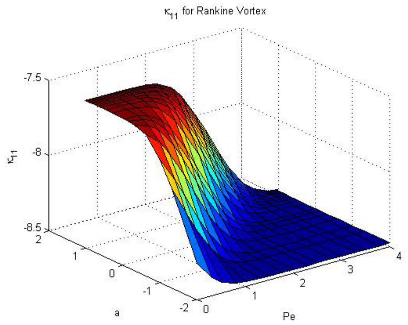

Fig. 1 (In $\log_{10}$ scale) K_{11}/Pe^2 vs flow parameters a and Pe.

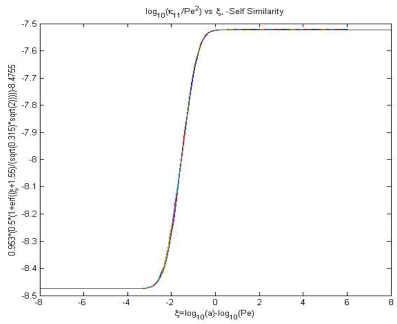

Fig. 2 Parametrization for $\log_{10}(K_{11}/\mathrm{Pe}^2)$.

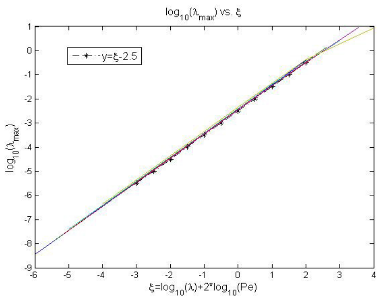

Fig. 3 Parametrization for $\lambda_{\max}$.

$\psi_p(x+x_p, y+y_p) = 12[\frac{((x+x_p)^2+(y+y_p)^2)^{3/2}}{9} - \frac{(x+x_p)^2+(y+y_p)^2}{6}]$, centered at the points (x_p, y_p) distributed according to a Poisson process with intensity λ. To isolate the effect of this initial randomization step on effective diffusivity, we take $\vec{V} = \vec{0}$. The relevant non-dimensional parameters are $\tilde{\lambda} = \lambda \cdot l_{\text{vor}}^2$ and $\mathrm{Pe}_{\text{loc}} = (l_{\text{vor}} \cdot v_{\text{vor}})/\kappa$.

3 Results

Under the assumption of scale separation, homogenization theory [4]-[7] tells us that enhancement in diffusion can be calculated through the solutions of the corresponding elliptic cell problems

$$\vec{U}(\vec{x}) \cdot \vec{\nabla}\chi(\vec{x}) + \mathrm{Pe}^{-1}\Delta\chi(\vec{x}) = -\vec{v}(\vec{x}) \tag{3}$$

with suitable boundary conditions for the corrector field $\chi(\vec{x})$, where $\vec{U}(\vec{x}) = \vec{V} + \vec{v}(\vec{x})$ for the first model and $\vec{U}(\vec{x}) = \vec{v}(\vec{x})$ for the second model. The solution $\chi(\vec{x})$ depicts how the velocity field distorts the passive scalar field on the small scale.

Using the corrector field $\chi(\vec{x})$, the effective diffusivity matrix K can be calculated through $K = \langle \vec{v} \otimes \vec{\chi} \rangle$, where $\langle \cdot \rangle$ denotes periodic average for the first model and an ensemble average for the second model.

In Fig.1, for the first model, we graph $\log_{10}(K_{11}/\mathrm{Pe}^2)$ as a function of flow parameters a and Pe in $\log_{10}$ scale. When we set the variable $\xi(a, \mathrm{Pe}) = \log_{10}(a) - \log_{10}(\mathrm{Pe})$ we can see the aforementioned data collapse in Fig.2 where we plot $\log_{10}(K_{11}/\mathrm{Pe}^2)$ vs ξ. We are able to obtain an explicit expression for the scaled effective diffusivity in terms of ξ as

$$\log_{10}(K_{11}/\mathrm{Pe}^2) = 0.953\left(0.5\left(1 + \operatorname{erf}\left(\frac{\xi + 1.55}{\sqrt{0.315}\sqrt{2}}\right)\right)\right) - 8.4755. \tag{4}$$

In Fig.3, for the second model, we give a parametrization of effective diffusivity in the direction of the maximum enhancement, given by the maximum eigenvalue $\lambda_{\max}$ of K, as a function of the single variable $\xi(\lambda, \mathrm{Pe}_{\text{loc}}) = \log_{10}(\lambda) + 2 * \log_{10}(\mathrm{Pe}_{\text{loc}})$ for the low Peclét number regime as $\lambda_{\max}(\xi) = \xi - 2.5$.

Acknowledgements The authors' work is supported by NSF grant OCE 0620956(CMG), and the first author's participation in the conference was supported by a SIAM travel grant.

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