

Introduction to Stochastic Differential Equations

MATH 4960/6960–1 – Fall 2007

Homework 3

Due Monday, November 19 at 5 PM

This homework has 200 points for MATP 6960 (260 points for MATP 4960) but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. If you score more than 100 points, the extra points do count toward your homework total.

1 MATP 4960 Problems

1.1 Inspection Frequency (30 points)

Suppose a machine, when started, operates properly for a random time T , which for simplicity we will take as an exponentially distributed random variable with mean τ_m . To determine whether the machine is working properly, a specialist needs to inspect it. Suppose the specialist checks the machine at regularly spaced deterministic times $\tau_i, 2\tau_i, 3\tau_i, \dots$ after the machine starts operating.

- a. **(10 points)** Calculate the probability distribution for the number of times the machine is inspected and found to be working properly. Express your answer in terms of the parameters τ_m and τ_i .
- b. **(10 points)** Suppose that the company running the machine incurs a cost of C_i per inspection and a cost of C_d per unit of time the machine is running improperly before being inspected (and therefore repaired). For example, if

we measure time in minutes then C_d is the cost per minute of improper operation. Calculate the expected cost incurred by the company from the time a machine starts operating until the first inspection which reveals that it is working improperly.

- c. **(10 points)** Suppose that when the machine is inspected and found to be working improperly, it is very quickly repaired (negligible time) and restarted, at which point it will again operate properly for a random time T which is exponentially distributed with mean τ_m and independent of any previous operating times. Show how the value of τ_i should be chosen to minimize the expected cost per time of running the machine in the long run.

1.2 Cumulants of Poisson Distribution (10 points)

Calculate the cumulants for a random variable X with a Poisson distribution:

$$\text{Prob}(X = j) = \frac{\lambda^j}{j!} e^{-\lambda} \text{ for } j = 0, 1, 2, \dots$$

1.3 Bad Batteries (10 points)

Suppose an underfunded facility is running equipment on very old batteries. Each of these batteries consists of N subcells, connected in series, each of which are supposed to produce voltage u . But each subcell has a probability p of failing, independently of each other, in which case that subcell produces no voltage. The voltage V in the battery is a sum of the voltages of its subcells, and the power P which the battery provides is V^2/R where R is a known deterministic constant (resistance). What is the average power produced by one of these batteries?

1.4 Continuous Random Walk (10 points)

Consider a random walk model on the real line rather than on the integers, so that the random walker starts at the origin, and at each epoch the random walker moves an amount given by a random variable with a prescribed probability distribution (which could be an arbitrary continuous, discrete, or hybrid distribution). The sign of the random variable indicates whether the random walker moves left or right. The steps taken by the random walker at each epoch are independent and identically distributed. Derive a general expression for the mean and variance of the position of the random walker after n epochs in terms of statistics of the prescribed probability distribution for each step.

2 MATP 4960/6960 Problems

2.1 Lebesgue Integration of Hybrid Random Variables (25 points)

Provide a rigorous proof for the statements made in class that if the probability distribution of a real-valued random variable X with CDF F_X is a combination of a discrete and absolutely continuous distribution, then the expectation of any bounded continuous function f of that random variable can be expressed as follows:

$$\mathbb{E}f(X) = \sum_{x_j \in \mathcal{A}(X)} a_j f(x_j) + \int_{-\infty}^{\infty} f(x) p^{(c)}(x) dx, \quad (1)$$

where $\mathcal{A}(X) = \{x_j\}$ is a countable set of atoms of the probability distribution, $a_j = \lim_{\epsilon \downarrow 0} F_X(x_j) - F_X(x_j - \epsilon)$ are the probability weights associated to the atoms, and $p^{(c)}(x) = \frac{d}{dx} F_X(x)$ for $x \notin \mathcal{A}(X)$ describes the absolutely continuous part of the distribution.

To make this formula rigorous, you need to evaluate $\mathbb{E}f(X)$ as a Lebesgue integral and show that you obtain Eq. (1). See Section 2.5 of Grigoriu, *Stochastic Calculus* and the lecture notes for the Lebesgue integration procedure. Your proof should involve a suitable approximation by simple functions. For simplicity, you may assume that $f \geq 0$; the extra steps needed to generalize to functions taking both signs is routine and tedious.

2.2 Fractional Occurrence of Dependent Events (20 points)

Consider a sequence $\{A_j\}_{j=1}^{\infty}$ of events (which need not be independent), and consider the random variable N_n defined as the number of events $\{A_j\}_{j=1}^n$ which occur in a given realization.

- (10 points) Express the mean and variance for $\frac{N_n}{n}$ in terms of a sum of probabilities involving the events $\{A_j\}_{j=1}^n$. To get a clean answer, approach this problem by expressing N_n in a simple way as a sum of indicator functions. This calculation should not involve complicated combinatorics or sums.
- (3 points) Under what conditions on the probabilities involving the events $\{A_j\}_{j=1}^n$ does the variance of $\frac{N_n}{n}$ approach zero (so that the fraction of events which occur obeys a weak law of large numbers)?
- (7 points) Provide a concrete probability model with events $\{A_j\}_{j=1}^{\infty}$ so that the condition derived in the previous part for the existence of a weak law of large numbers is satisfied.

2.3 Multivariate Cumulants (10 points)

Cumulants and generating functions can readily be extended to collections of random variables, which as usual in this course I will represent with a vector: $\mathbf{X} = [X_1, X_2, \dots, X_N]$. First we define the characteristic function (moment generating function) for this collection of random variables through

$$\phi_{\mathbf{X}}(\mathbf{k}) = \langle e^{i\mathbf{k}\cdot\mathbf{X}} \rangle$$

and the cumulant generating function by

$$\ln \phi_{\mathbf{X}}(\mathbf{k}).$$

Here the argument of these functions \mathbf{k} is a vector with the same dimensionality (N) as the random variable \mathbf{X} . Multivariate moments can be calculated from the characteristic function through the relation:

$$\langle X_1^{n_1} X_2^{n_2} \dots X_N^{n_N} \rangle = \left(-i \frac{\partial}{\partial k_1} \right)^{n_1} \left(-i \frac{\partial}{\partial k_2} \right)^{n_2} \dots \left(-i \frac{\partial}{\partial k_N} \right)^{n_N} \phi_{\mathbf{X}}(\mathbf{k}) \Big|_{\mathbf{k}=\mathbf{0}},$$

and the corresponding cumulants are defined by analogous differentiation of the cumulant generating function:

$$\langle\langle X_1^{n_1} X_2^{n_2} \dots X_N^{n_N} \rangle\rangle = \left(-i \frac{\partial}{\partial k_1} \right)^{n_1} \left(-i \frac{\partial}{\partial k_2} \right)^{n_2} \dots \left(-i \frac{\partial}{\partial k_N} \right)^{n_N} \ln \phi_{\mathbf{X}}(\mathbf{k}) \Big|_{\mathbf{k}=\mathbf{0}},$$

Here all the exponents $\{n_j\}_{j=1}^N$ are understood to be nonnegative integers.

Show that if the random variables $\{X_j\}_{j=1}^N$ are independent (but not necessarily identically distributed), then

$$\langle\langle X_1^{n_1} X_2^{n_2} \dots X_N^{n_N} \rangle\rangle = 0 \text{ whenever } n_j \geq 1 \text{ and } n_{j'} \geq 1 \text{ for some } 1 \leq j < j' \leq N.$$

This reflects the property that cumulants remove redundant information from moments. That is, multivariate cumulants of independent random variables vanish whenever more than one of the variables is involved because independence implies there is no further information available from considering the random variables two or more at a time than there was from considering them one at a time.

2.4 Smooth Operation (15 points)

Prove that if X is an absolutely continuous real-valued random variable and Y is a real-valued random variable with arbitrary probability distribution, and X and Y are independent, then $X + Y$ is a random variable with absolutely continuous probability distribution.

2.5 Equipartition Theorem in Microcanonical Ensemble (25 points)

Consider a system of two noninteracting particles with constant masses m_1 and m_2 ; their Hamiltonian is then given simply by the kinetic energy: $H(p_1, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$. (Unfortunately, there is a bit of notation clash – momenta are also denoted by a lower case p in mechanics. If you find it distracting, you may shift notation in your solution.) In the microcanonical ensemble in statistical mechanics, the probability distribution for the momenta $\mathbf{P} = (P_1, P_2)$ of these variables in an isolated system with energy E is a singular distribution which can be described in terms of a generalized PDF

$$p_{\mathbf{P}}(p_1, p_2) = \frac{\delta(H(p_1, p_2) - E)}{Z}$$

where Z is a normalization constant so that

$$\int_{\mathbb{R}^2} p_{\mathbf{P}}(p_1, p_2) dp_1 dp_2 = 1.$$

- (20 points) Compute the marginal probability distribution for P_1 , along with the mean momentum $\mathbb{E}P_1$ and mean energy $\mathbb{E}\frac{P_1^2}{2m_1}$ of the first particle.
- (5 points) Your results should agree with the equipartition theorem from statistical mechanics which states that each particle should have the same average energy (despite different physical properties, such as mass). Show that if we would have defined the microcanonical ensemble as a uniform distribution on the energy surface $H(p_1, p_2) = E$, this equipartition theorem result would be violated.

2.6 Spherical Galaxy with Multiple Stars (25 points)

Revisit Problem 2.4 from Homework 2, but now take into account the presence of n stars independently and uniformly distributed throughout the sphere.

- (15 points) Plot the PDF for the total gravitational force exerted along a specified direction $\hat{\mathbf{e}}$ at the center of the sphere for galaxies with $n = 2, 5, 10, 1000$, and 10^6 stars.
- (5 points) Again plot the PDF for the total gravitational force exerted along a specified direction $\hat{\mathbf{e}}$ at the center of the sphere for galaxies with $n = 2, 5, 10, 1000$, and 10^6 stars, but now adjust the radius R of the galaxy so that the average density of stars remains constant as you increase n .
- (5 points) Comment on your results.

2.7 How Much Trying to Succeed in Business (20 points)

Suppose you are trying to raise some capital (\$) for some business enterprise. You pursue a sequence of fund-raising “endeavors” which are all of the same type (meeting venture capitalists, hitting up relatives, bake sales, robbing stores, . . .); hopefully your business idea is more creative than your fundraising approach. You direct your fund-raising endeavors so that they do not interfere with each other. Consequently, you can model the amount of funds raised with each endeavor as a nonnegative random variable (neglecting expenses incurred with each endeavor) such that the amount of funds raised in separate endeavors may be considered as independent and identically distributed. Your objective is to raise a fixed amount of funds. Based on the probability modeling assumptions described, derive an expression for the probability distribution for the number of fund-raising endeavors needed to raise the desired amount of funds. Your answer should involve both the target level of funds as well as the statistical properties of the amount of funds raised in each endeavor.

For full credit, you should allow the random variable describing the amount of funds raised per endeavor to have both discrete and continuous components. Of course, technically speaking, money always takes discrete values when measured in dollars and cents, but most financial models in practice smooth over the discretization of cents and work with continuous random variables. But in this case, a discrete component might be included which represents special values that one might sometimes obtain exactly (an asking price, a target, or some nice round number).

2.8 Surfing over Loss (15 points)

Show that the probability that a symmetric random walk (with steps ± 1 equally likely) will return to the starting point at epoch $2n$ and never have assumed negative values up to that point is the same as $2f_{2n+2}$, twice the probability that the random walk returns to its starting point for the first time at epoch $2n + 2$.

2.9 Stumbling Walk (20 points)

Consider a asymmetric random walk with independent, identically distributed steps at each epoch, with probability p to move right one unit, probability q to move left one unit, and probability $1 - p - q$ to remain stationary during the epoch. In class we developed several results about the path properties of random walks when $p + q = 1$; you are asked here to derive more general expressions for the case $0 < p, q < p + q \leq 1$ for the following quantities:

- a. (5 points) the probability that the random walk ever reaches the value $r > 0$,

- b. **(5 points)** the mean number of epochs until the random walk reaches the value $r > 0$ for the first time,
- c. **(5 points)** the probability that the random walk ever returns to its starting point, and
- d. **(5 points)** the mean number of epochs required before returning to the starting point for the first time.

Comment on how your results differ from the $p + q = 1$ formulas, and offer intuitive explanations when possible.

2.10 Later Passage Time (25 points)

Consider an asymmetric walk without the possibility to stand still ($p + q = 1$ in the notation of the previous problem).

- a. **(5 points)** Provide a mathematical definition for a random variable which returns the epoch at which the random walk achieves a value $r > 0$ for the k th time. (If the value is not achieved k times, the random variable may be taken to take the value ∞ .)
- b. **(10 points)** Derive an explicit expression for the probability generating function for this random variable.
- c. **(10 points)** Develop an expression for the probability distribution for the last time the random walk visits the value $r > 0$ in terms of the probability generating function constructed in Part (b).