

Singular Perturbation Theory for Homoclinic Orbits in a Class of Near-Integrable Dissipative Systems *

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Abstract

This paper presents a new unified theory of orbits homoclinic to resonance bands in a class of near-integrable dissipative systems. It describes three sets of conditions, each of which implies the existence of homoclinic or heteroclinic orbits that connect equilibria or periodic orbits in a resonance band. These homoclinic and heteroclinic orbits are born under a given small dissipative perturbation out of a family of heteroclinic orbits that connect pairs of points on a circle of equilibria in the phase space of the nearby integrable system. The result is a constructive method that may be used to ascertain the existence of orbits homoclinic to objects in a resonance band, as well as to determine their precise shape, asymptotic behavior, and bifurcations in a given example. The method is a combination of the Melnikov method and geometric singular perturbation theory for ordinary differential equations.

1 Introduction

Completely integrable Hamiltonian systems are a fairly rare occurrence. Nevertheless, since they can be solved explicitly, they are the first step in the description of many fundamental physical phenomena, such as the motion of rigid bodies or the circling of the earth around the sun. Knowledge of the phase-space properties of these special idealized systems is used in conjunction with perturbation theory to describe more realistic physical problems, for instance, problems that exhibit irregular or chaotic behavior.

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The perturbation method most commonly used to show the presence of chaotic dynamics in near-integrable systems is the Melnikov method. First developed for time-periodically perturbed planar systems [1–5], it was soon generalized to cover multi-degree-of-freedom systems as well [6–17]. This method is particularly convenient for multi-dimensional Hamiltonian systems, where it combines with the KAM theory [18–22] to yield the existence of Smale-horseshoe chaos and Arnold diffusion in many problems [7–9, 23–29].

The use of the multi-dimensional Melnikov method for near-integrable dissipative systems is restricted to special cases [13, 14, 16, 30, 31]. In all of these cases, averaging or some singular perturbation method must be used together with the Melnikov method in order to show the existence of some homoclinic or heteroclinic orbits, whose presence causes nearby phase points to behave chaotically. The singular perturbation aspect is stressed the most in ref. [30]. That paper shows how to construct a spiral-saddle connection out of a circle of equilibria and a two-dimensional surface of heteroclinic orbits connecting certain pairs of points on this circle. The spiral-saddle itself, as well as the spiraling part of the connection are born out of the circle of equilibria under perturbation; hence the singular nature of the problem.

This paper presents a geometric theory of phenomena that can emerge under perturbation out of a manifold of orbits homoclinic to a circle of equilibria, which lies on an unstable invariant annulus in the phase space of an $(n + 1)$ -degree-of-freedom integrable dynamical system. In particular, the paper discusses a number of possible homoclinic and heteroclinic orbits connecting equilibria and periodic orbits that lie inside a resonance band [32–34] created by the perturbation out of the circle of equilibria.

The main results of this paper are presented in the three theorems in section 3. These theorems describe the various geometric situations that give rise to homoclinic or heteroclinic orbits connecting objects in a resonance band. Theorem 1 describes orbits that result from intersections of $(n + 1)$ -dimensional stable and $(n + 1)$ -dimensional unstable manifolds; theorem 2 describes orbits that result from intersections of $(n + 2)$ -dimensional stable and $(n + 1)$ -dimensional unstable manifolds; and theorem 3 describes orbits that result from intersections of $(n + 2)$ -dimensional stable and n -dimensional unstable manifolds.

The result presented in ref. [30] is a special example of this paper’s theorem 3. A very similar example is given in ref. [31]. An example of theorem 1 is presented in ref. [35]. This example shows how to construct orbits homoclinic to saddles in a resonance band that have purely real eigenvalues. A similar example is studied in ref. [36], in which an independent

proof is given for a special case of this paper's theorem 1, and careful numerical calculations are performed that support the theoretical findings.

The proofs of the three main theorems of this paper that are described in section 7 require a fair amount of background, which is outlined in sections 4, 5, and 6. In particular, all three proofs follow virtually the same geometric idea that consists of three main steps. The first step is outlined in section 4. In this step, an unperturbed unstable invariant annulus, which contains a circle of equilibria and is connected to itself by an $(n + 2)$ -dimensional homoclinic manifold, is shown to persist under perturbation, and the Melnikov method is used to ascertain whether a two-dimensional homoclinic manifold of orbits that are biasymptotic to this persisting annulus exists. In the second step, rescaling is used to describe the dynamics in the resonance band that is created by the perturbation on the persisting annulus out of the unperturbed circle of equilibria. This step is developed in section 5. In the third step, geometric singular perturbation theory [37, 38] is used to connect the homoclinic dynamics, which are transverse to the persisting annulus, to the dynamics along this annulus in order to describe the precise asymptotic behavior of the homoclinic orbits. This step is carried out in sections 6 and 7.

A Hamiltonian counterpart of this paper was developed in refs. [39–41]. For two-degree-of-freedom systems, the Hamiltonian result is very general, because the Melnikov function used in that situation can be computed explicitly as an energy difference. The details of the Hamiltonian case have much in common with theorem 1 of this paper. In particular, two crucial stepping stones in its proof are propositions 7.1 and 7.2. However, the geometry of the Hamiltonian case is very different from the geometry described in the present paper. Namely, in the Hamiltonian case, the result describes two-dimensional surfaces of orbits homoclinic to nested families of periodic orbits as opposed to isolated homoclinic or heteroclinic orbits described in this paper. Moreover, orbits homoclinic to objects inside a resonance band are generic in the Hamiltonian case, but only occur on lower-dimensional submanifolds of the parameter space in most subcases of the dissipative case presented here.

Methods for finding orbits homoclinic to resonance bands may be applied to some of the systems that have undergone a change of variables into a frame rotating with the same frequency as an external force and subsequent averaging. Examples of such systems are [42] in the theory of Josephson's junctions; [25] in nonlinear fiber optics; [26, 28, 29] in laser-matter interaction; [13, 23, 24, 31, 43] in the theory of water waves; [27] in the theory of vibrating plates. The advantage of the methods presented in this paper's theorems 1, 2, and 3, as well as in the Main Theorem of ref. [40], is that their hypotheses are easily verified

in specific situations. In fact, their verification requires only algebraic manipulations. This makes the method described in this paper a potentially powerful tool for solving physical and engineering problems.

This paper is organized as follows. In section 2, the problem of orbits homoclinic to resonance bands is set up. In section 3, the main results of the paper are stated. In section 4, results from persistence theory of normally hyperbolic invariant manifolds that are needed for the understanding of this paper are discussed, and a brief review of the multi-dimensional Melnikov method is given. In section 5, an approach to analyzing resonance bands is explained. In section 6, geometric singular perturbation theory is used to calculate local stable and unstable manifolds of objects in a resonance band. In section 7, the three main theorems of this paper are proven. Finally, in section 8, a simple example is shown to satisfy the conditions of the three main theorems for certain parameter values.

2 The Setup

We consider systems of the form

$$\dot{x} = JD_x H(x, I) + \varepsilon g^x(x, I, \theta, \lambda), \quad (2.1a)$$

$$\dot{I} = \varepsilon g^I(x, I, \theta, \lambda), \quad (2.1b)$$

$$\dot{\theta} = \Omega(x, I) + \varepsilon g^\theta(x, I, \theta, \lambda). \quad (2.1c)$$

Here $x = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$, $I \in \mathbb{R}$ and $\theta \in S^1$, D_x denotes the partial derivatives with respect to x , $\lambda \in \mathbb{R}$ is a real parameter, $\varepsilon \ll 1$ is a small parameter, and $J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$, with Id being the $n \times n$ identity matrix.

When we set $\varepsilon = 0$, we obtain the unperturbed system:

$$\dot{x} = JD_x H(x, I), \quad (2.2a)$$

$$\dot{I} = 0, \quad (2.2b)$$

$$\dot{\theta} = \Omega(x, I). \quad (2.2c)$$

We immediately note that equation (2.2a) is a one-parameter family of Hamiltonian systems for the variable x and can be analyzed independently of θ . Equation (2.2c) can be solved by quadrature, once equation (2.2a) has been solved.

We first make two assumptions about the system (2.2). The first assumption concerns its solvability:

Assumption 1 *For all I with $I_1 < I < I_2$, for some I_1 and I_2 , the system (2.2a) is completely integrable; that is, there exists a smooth family of n integrals of motion, $K_1(x, I) = H(x, I)$, $K_2(x, I)$, \dots , $K_n(x, I)$, whose gradients $D_x K_1(x, I)$, $D_x K_2(x, I)$, \dots , $D_x K_n(x, I)$ are linearly independent at all points x which are not equilibria of (2.2a) and pairwise satisfy the relationship*

$$\langle JD_x K_i(x, I), D_x K_j(x, I) \rangle = 0, \quad (2.3)$$

for all $i, j = 1, \dots, n$.

This assumption implies that, at least in principle, solutions to equation (2.2a) may be obtained by quadratures; see for instance [17].

The second assumption introduces homoclinic orbits into the phase space of equations (2.2a):

Assumption 2 *For every I with $I_1 < I < I_2$, equation (2.2a) possesses a hyperbolic equilibrium $x = X(I)$, which varies smoothly with I and a manifold $W(X(I))$ of homoclinic orbits, connecting the equilibrium at $x = X(I)$ to itself.*

We remark that the stable and unstable manifolds $W^s(X(I))$ and $W^u(X(I))$ of the equilibrium $X(I)$ must both be n -dimensional, since the eigenvalues of the matrix $JD_x^2 H(X(I), I)$ come in pairs $\kappa, -\kappa$. The homoclinic manifold $W(X(I))$ must also be n -dimensional because of the linear independence of the gradients $D_x K_1(x, I)$, \dots , $D_x K_n(x, I)$. (See, for instance, [17, proposition 4.1.3].)

Since the system (2.2a) is autonomous, all the solutions on the homoclinic manifold $W(X(I))$ can be represented in the form $x^h(t - t_0, I, \phi)$, where $\phi \in \mathbb{R}^{n-1}$ is a vector of parameters. A consistent parametrization of individual orbits in the manifold $W(X(I))$ can be obtained by setting $t_0 = 0$ and varying t .

In the full $(2n + 2)$ -dimensional phase space of the system (2.2), the family of equilibria at $x = X(I)$ forms a two-dimensional invariant annulus \mathcal{M} foliated by periodic orbits with coordinates $x = X(I)$, I , and $\theta = \Omega(X(I), I)t + \theta_0$, with $I_1 < I < I_2$. The annulus \mathcal{M} possesses $(n + 2)$ -dimensional stable and unstable manifolds $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$, which are the unions over the interval $I_1 < I < I_2$ of the cartesian products of the manifolds $W^s(X(I))$

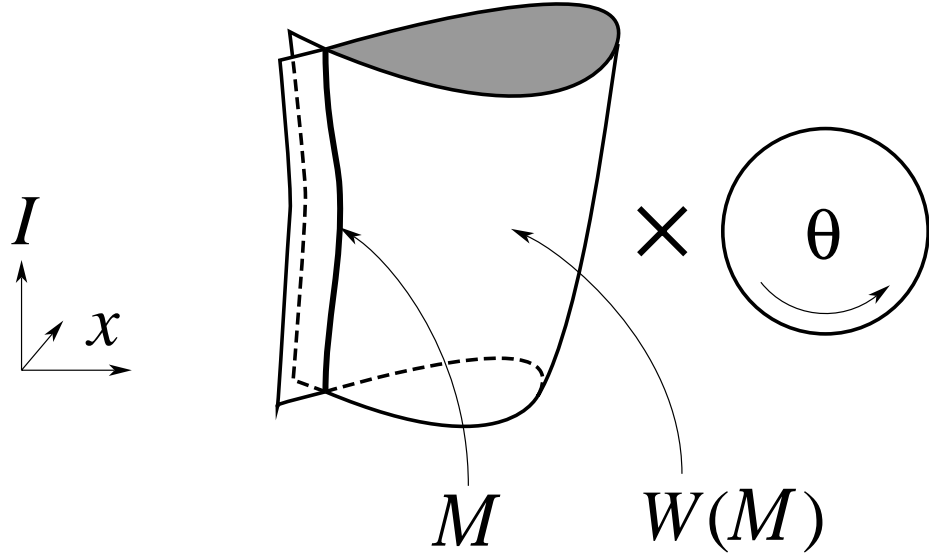


Figure 1: The invariant annulus \mathcal{M} and its three-dimensional homoclinic manifold $W(\mathcal{M})$ are the cartesian product of a circle with a curve segment filled with equilibria, and its two-dimensional homoclinic manifold.

and $W^u(X(I))$ with the angle θ , respectively. The manifolds $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ intersect along the $(n + 2)$ -dimensional homoclinic manifold $W(\mathcal{M})$, which is the union over the interval $I_1 < I < I_2$ of the cartesian products of the homoclinic manifolds $W(X(I))$ with the angle θ , shown in figure 1. We remark that the homoclinic manifold $W(\mathcal{M})$ can be parametrized by t , I , ϕ , and θ_0 in the representation:

$$x = x^h(t, I, \phi), \quad I = I, \quad \theta = \theta^h(t, I, \phi) + \theta_0, \quad (2.4)$$

with

$$\theta^h(t, I, \phi) = \int_0^t \Omega(x^h(s, I, \phi), I) ds.$$

It can also be represented implicitly by the set of equations:

$$K_i(x, I) - K_i(X(I), I) = 0, \quad i = 1, \dots, n, \quad (2.5)$$

which hold on the annulus \mathcal{M} at $x = X(I)$ and, therefore, also on the homoclinic manifold $W(\mathcal{M})$.

In order to study orbits homoclinic to resonance bands, we make the assumption:

Assumption 3 For some I_0 with $I_1 < I_0 < I_2$ we have

$$\Omega(X(I_0), I_0) = 0$$

with

$$\frac{d\Omega(X(I_0), I_0)}{dI} \neq 0.$$

This assumption implies that for $I = I_0$, the frequency of the periodic orbit on the annulus \mathcal{M} passes through a simple zero, so that this periodic orbit is really a circle of equilibria. As will be shown in section 5, this circle will break up under the given perturbation into a *resonance band*, which is the main object of this study.

Any equilibrium p on the circle is determined by its value of the angle $\theta = \theta(p)$. The unstable manifold of the point p is the set parametrized by the variables t and ϕ in the formulas $x = x^h(t, I_0, \phi)$, $I = I_0$, $\theta = \theta^h(t, I_0, \phi) + \theta_0(p, \phi)$ with the phase angle $\theta_0(p, \phi)$ defined by $\theta_0(p, \phi) = \theta(p) - \theta^h(-\infty, I_0, \phi)$. This set is an n -dimensional manifold, in general foliated by heteroclinic connections between p and other equilibria on the circle, whose θ coordinates are given by the formula $\theta = \theta(p) + \Delta\theta(\phi)$, in which the expression $\Delta\theta(\phi) = \theta^h(\infty, I_0, \phi) - \theta^h(-\infty, I_0, \phi)$ depends only on the value of the parameter vector ϕ and not on the initial equilibrium p , because of the phase symmetry of equations (2.2) in θ . Similar statements hold for the stable manifold $W^s(p)$.

As mentioned above, the circle of equilibria at $I = I_0$ breaks up under the perturbation into a resonance band. In order for this resonance band to contain only a finite number of discrete equilibria, we assume

Assumption 4 *At any fixed value of the parameter λ , the function $g^I(X(I_0), I_0, \theta, \lambda)$ has only finitely many simple zeros in θ for $0 \leq \theta \leq 2\pi$.*

Finally, we define the Melnikov vector, $\mathbf{M}(I, \phi, \theta_0, \lambda)$, whose n components $M_i(I, \phi, \theta_0, \lambda)$ are given by the formulas

$$M_i(I, \phi, \theta_0, \lambda) = \int_{-\infty}^{\infty} \langle \mathbf{n}_i, g \rangle dt, \quad (2.6)$$

where

$$\begin{aligned} \mathbf{n}_i &= \left(D_x K_i(x^h(t, I, \phi), I), D_I K_i(x^h(t, I, \phi), I) - \frac{dK_i}{dI}(X(I), I), 0 \right) \\ &= \left(D_x K_i(x^h(t, I, \phi), I), D_I K_i(x^h(t, I, \phi), I) - D_I K_i(X(I), I), 0 \right), \end{aligned}$$

for $i = 1, \dots, n$, are the n normals to the homoclinic manifold $W(\mathcal{M})$ that can be calculated from the equation (2.5), and $g = (g^x, g^I, g^\theta)$ is the $\mathcal{O}(\varepsilon)$ perturbation part of the vector field

(2.1), calculated at $x = x^h(t, I, \phi)$, I and $\theta = \theta^h(t, I, \phi) + \theta_0$; see [1–17]. The above two expressions for the normal \mathbf{n}_i are equivalent because $D_x K_i(X(I), I) = 0$. (This follows from differentiating equation (2.3) with $j = 1$ upon x , substituting $x = X(I)$, and remembering that $x = X(I)$ is a hyperbolic equilibrium of the equation (2.2a), so that the matrix $JD^2H(X(I), I)$ is invertible; see [10] or [17, p. 407].)

For the rest of this paper, we assume

Assumption 5 *For $I = I_0$ and some $\phi = \bar{\phi}$, $\theta_0 = \bar{\theta}_0$ and $\lambda = \bar{\lambda}$ the following two statements are true:*

1. $\mathbf{M}(I_0, \bar{\phi}, \bar{\theta}_0, \bar{\lambda}) = 0$,
2. $D_{(\phi, \theta_0)}\mathbf{M}(I_0, \bar{\phi}, \bar{\theta}_0, \bar{\lambda})$ has maximal rank.

3 The Main Results

In this section, we state the main results of this paper. They are described in theorems 1, 2, and 3, and follow from a series of preliminary results which we outline next. All the proofs and further details are relegated to sections 4, 5, 6, and 7.

First, we observe that the results of Fenichel [44–46] imply that the annulus \mathcal{M} and its stable and unstable manifolds $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ persist under perturbation as a locally invariant annulus \mathcal{M}_ε and its stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$. The precise nature of these manifolds will be discussed in proposition 4.1. What is important for this outline is that the perturbed annulus \mathcal{M}_ε can be written as a graph over the I and θ variables in the form

$$x = X_\varepsilon(I, \theta, \lambda, \varepsilon) \tag{3.1}$$

for some smooth function $X_\varepsilon(I, \theta, \lambda, \varepsilon)$ with $X_0(I, \theta, \lambda, 0) = X(I)$.

The circle of equilibria that exists on the unperturbed annulus \mathcal{M} breaks up under perturbation into a *resonance band* lying on the perturbed annulus \mathcal{M}_ε . This resonance band is best described in the following way: We restrict the dynamics of equations (2.1) to the annulus \mathcal{M}_ε using formula (3.1). Following [32–34], we then “blow up” the region near $I = I_0$ using the transformation $I = I_0 + \sqrt{\varepsilon}h$, rescale time using $\tau = \sqrt{\varepsilon}t$, and Taylor

expand in $\sqrt{\varepsilon}$, to obtain the equations

$$h' = g^I(X(I_0), I_0, \theta, \lambda) + \mathcal{O}(\sqrt{\varepsilon}), \quad (3.2a)$$

$$\theta' = \frac{d\Omega}{dI}(X(I_0), I_0) h + \mathcal{O}(\sqrt{\varepsilon}), \quad (3.2b)$$

with $' = \frac{d}{d\tau}$. Higher order terms in these equations can be easily computed just in terms of differentiations and algebraic operations alone, as implied by proposition 5.1.

In the limit as $\varepsilon \rightarrow 0$, we obtain the *rescaled* or *outer system*

$$h' = g^I(X(I_0), I_0, \theta, \lambda), \quad (3.3a)$$

$$\theta' = \frac{d\Omega}{dI}(X(I_0), I_0) h. \quad (3.3b)$$

The outer system (3.3) can be derived from the *rescaled Hamiltonian*

$$\mathcal{H}(h, \theta, \lambda) = \frac{1}{2} \frac{d\Omega}{dI}(X(I_0), I_0) h^2 + V(\theta, \lambda), \quad (3.4)$$

with

$$V(\theta, \lambda) = - \int_0^\theta g^I(X(I_0), I_0, s, \lambda) ds,$$

via the canonical formulas

$$h' = -D_\theta \mathcal{H}(h, \theta, \lambda), \quad \theta' = D_h \mathcal{H}(h, \theta, \lambda).$$

Note that the limiting outer system (3.3) is Hamiltonian also when systems (3.2), (2.1), and even (2.2) are not.

System (3.2) can be investigated with the help of system (3.3) by a mixture of phase-plane and perturbation techniques. One approach to this investigation is outlined in section 5. The phase portraits of a typical outer system (3.3) and its perturbed counterpart (3.2) are shown in figure 2.

In order to investigate orbits homoclinic or heteroclinic to possible equilibria and periodic orbits of equations (3.2) in the full $x - I - \theta$ phase space, we simply set $I = I_0 + \sqrt{\varepsilon}h$ in equations (2.1) and let $\varepsilon \rightarrow 0$. The resulting system is the *inner system*

$$\dot{x} = JD_x H(x, I_0), \quad (3.5a)$$

$$\dot{h} = 0, \quad (3.5b)$$

$$\dot{\theta} = \Omega(x, I_0). \quad (3.5c)$$

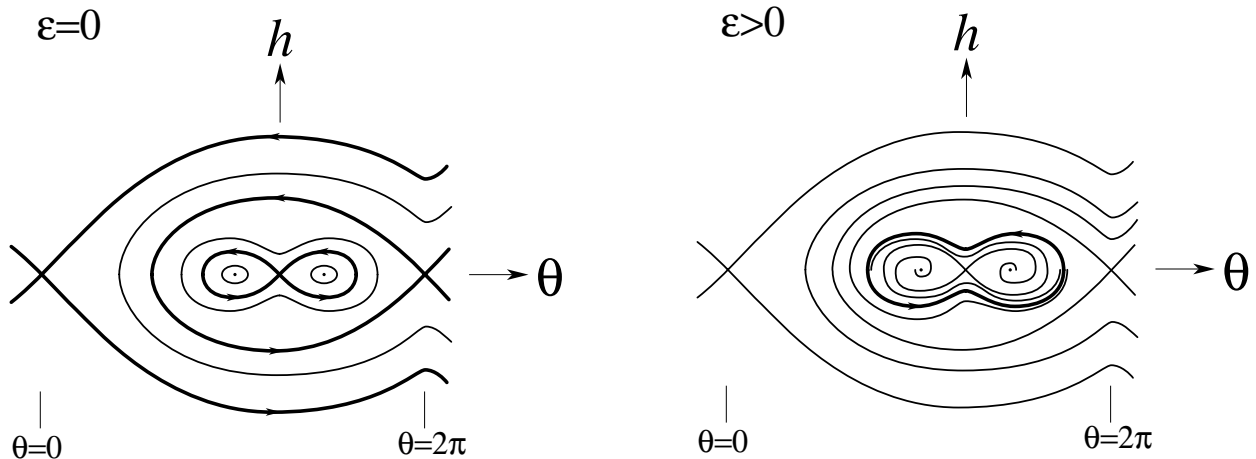


Figure 2: A typical phase portrait of a rescaled or outer system for $\varepsilon = 0$ and $\varepsilon > 0$. All the points whose θ coordinates differ by a multiple of 2π must be identified.

This system is a singular limit of equations (2.1) in the sense that the circle of equilibria for the unperturbed equations (2.2) at $I = I_0$, $x = X(I_0)$, $0 \leq \theta \leq 2\pi$ has been “blown up” into a cylinder of equilibria with $x = X(I_0)$, $0 \leq \theta \leq 2\pi$ and arbitrary h . Equilibria on this cylinder are unstable, and the structure of each circle $h = \text{constant}$ and the heteroclinic orbits that connect pairs of points on it is the same (including the solutions on the heteroclinic orbits) as the structure of the circle at $I = I_0$ and its homoclinic manifold $W(X(I_0))$. Thus, the $h - \theta$ cylinder of equilibria at $x = X(I_0)$ is connected to itself by an $(n + 2)$ -dimensional homoclinic manifold.

In what is to follow, we confine the values of h to the interval $-C < h < C$ with some large enough constant C . The annular portion of the $h - \theta$ cylinder between the circles $h = -C$ and $h = C$ will be called $\hat{\mathcal{M}}$. The annulus $\hat{\mathcal{M}}$ and its stable and unstable manifolds $W^s(\hat{\mathcal{M}})$ and $W^u(\hat{\mathcal{M}})$ are the limits as $\varepsilon \rightarrow 0$ of a nearby annulus $\hat{\mathcal{M}}_\varepsilon$ and its stable and unstable manifolds, $W^s(\hat{\mathcal{M}}_\varepsilon)$ and $W^u(\hat{\mathcal{M}}_\varepsilon)$. In the original $x - I - \theta$ coordinates, the annulus $\hat{\mathcal{M}}_\varepsilon$ and its stable and unstable manifolds $W^s(\hat{\mathcal{M}}_\varepsilon)$ and $W^u(\hat{\mathcal{M}}_\varepsilon)$ are just small pieces of the perturbed annulus \mathcal{M}_ε and the manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$, which shrink to zero like $\sqrt{\varepsilon}$ as $\varepsilon \rightarrow 0$.

The inner and outer systems are complementary in the following way: The inner system describes the structure of homoclinic orbits away from the annulus $\hat{\mathcal{M}}$, but the annulus $\hat{\mathcal{M}}$ itself consists of equilibria, and all the nontrivial dynamics on the nearby annulus $\hat{\mathcal{M}}_\varepsilon$ are lost in the inner limit. On the other hand, the $h - \theta$ cylinder $\hat{\mathcal{M}}$ for the outer equations (3.3) possesses nontrivial dynamics due to the rescaling of time; however, all the dynamics away

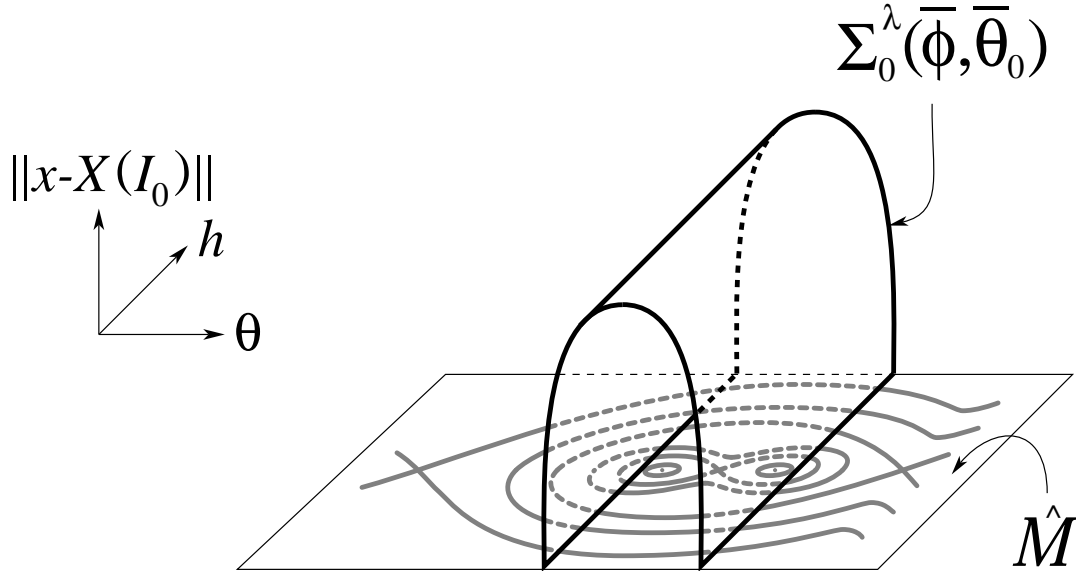


Figure 3: The limiting homoclinic manifold $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ connects equilibria that lie on the line $\theta = \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})$ to those that lie on the line $\theta = \bar{\theta}_0 + \Delta\theta_+(\bar{\phi})$ on the annulus $\hat{\mathcal{M}}$. Gray curves on $\hat{\mathcal{M}}$ represent the orbit structure on this annulus under the rescaled or outer system

from $x = X(I_0)$ are lost in this system. This situation is typical of singular perturbation problems in which we combine the information obtained from systems (3.5) and (3.3) to obtain useful information about the behavior of system (2.1) near the resonance band on the perturbed annulus \mathcal{M}_ε at $I = I_0$.

We mentioned above that the annulus $\hat{\mathcal{M}}$ in the inner limit possesses an $(n + 2)$ -dimensional homoclinic manifold. We will show in proposition 4.3 that a two-dimensional intersection surface, $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$, of the manifolds $W^s(\hat{\mathcal{M}}_\varepsilon)$ and $W^u(\hat{\mathcal{M}}_\varepsilon)$ survives from this homoclinic manifold for nonzero ε . This surface corresponds to the transverse zero of the Melnikov vector at $I = I_0$, $\phi = \bar{\phi}$, $\theta_0 = \bar{\theta}_0$, and $\lambda = \bar{\lambda}$, whose existence was assumed in assumption 5. The surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ exists for all λ close enough to $\lambda = \bar{\lambda}$. In the inner limit, the surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ tends to the *limiting homoclinic intersection surface* $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$, shown in figure 3. This surface consists of those heteroclinic orbits connecting equilibria on the cylinder $\hat{\mathcal{M}}$ whose ϕ and θ_0 parameters equal $\phi = \bar{\phi}(I_0, \lambda)$, $\theta_0 = \bar{\theta}_0(I_0, \lambda)$, where $\bar{\phi}(I, \lambda)$ and $\bar{\theta}_0(I, \lambda)$ are two smooth functions, defined for I and λ near $I = I_0$ and $\lambda = \bar{\lambda}$, with $\bar{\phi}(I_0, \bar{\lambda}) = \bar{\phi}$ and $\bar{\theta}_0(I_0, \bar{\lambda}) = \bar{\theta}_0$, that identically satisfy the equation $\mathbf{M}(I, \bar{\phi}(I, \lambda), \bar{\theta}_0(I, \lambda), \lambda) = 0$. Orbits on the limiting intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ are thus explicitly given by the formulas $x = x^h(t, I_0, \bar{\phi}(I_0, \lambda))$, $h = h$, $\theta = \theta^h(t, I_0, \bar{\phi}(I_0, \lambda)) + \bar{\theta}_0(I_0, \lambda)$. Heteroclinic or-

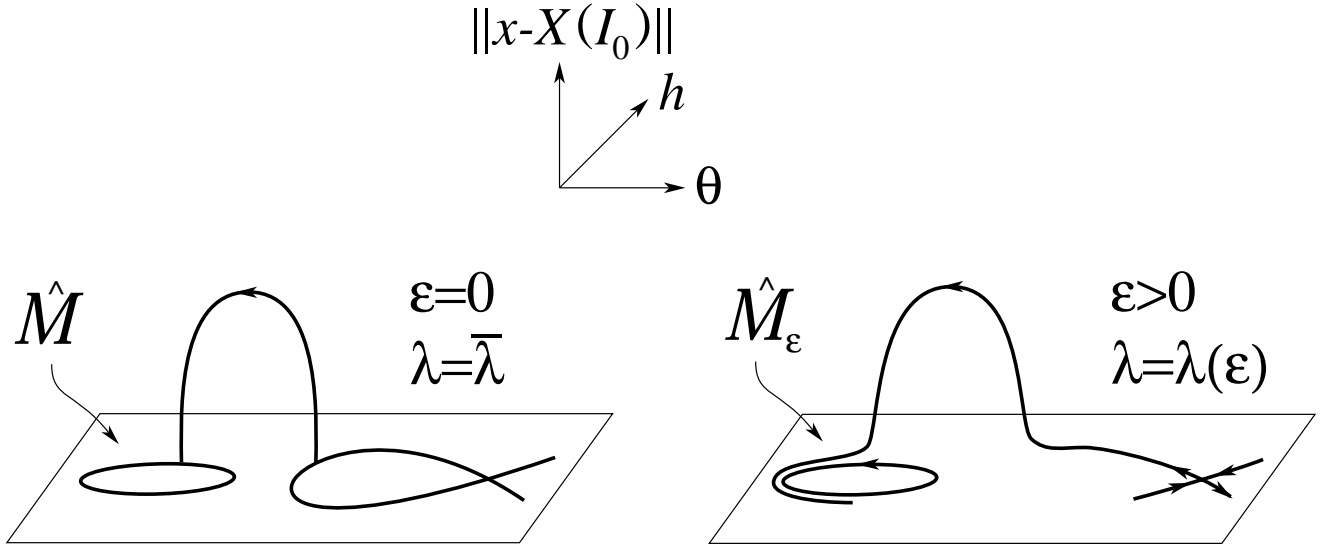


Figure 4: The situation discussed in theorem 1 where the curve $O_{1,\varepsilon}(\lambda)$ is the restricted unstable manifold of the saddle $s_\varepsilon(\lambda)$ and the curve $O_{2,\varepsilon}(\lambda)$ is an unstable limit cycle on the annulus $\hat{\mathcal{M}}_\varepsilon$. In this case, a heteroclinic orbit connects the saddle $s_\varepsilon(\lambda(\varepsilon))$ to the periodic orbit $O_{2,\varepsilon}(\lambda(\varepsilon))$ for some $\lambda = \lambda(\varepsilon)$.

bits comprising the surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ emerge from the $h - \theta$ cylinder $\hat{\mathcal{M}}$ along the line $\theta = \bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))$ and return to $\hat{\mathcal{M}}$ along the line $\theta = \bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda))$, where

$$\Delta\theta_+(\phi) = \int_0^\infty \Omega(x^h(s, I_0, \phi), I_0) ds, \quad \Delta\theta_-(\phi) = \int_{-\infty}^0 \Omega(x^h(s, I_0, \phi), I_0) ds. \quad (3.6)$$

We are now ready to state the three main theorems of this paper. All three concern special orbits on the intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$. These orbits are homoclinic or heteroclinic connections between equilibria and periodic orbits in the resonance band, and arise in three different geometric situations. Orbits described in theorems 1 and 3 exist for discrete values of the parameter λ , whereas orbits described in theorem 2 exist on intervals of λ .

The first special situation that leads to the existence of homoclinic or heteroclinic orbits occurs when there exist two families of curves $O_{1,\varepsilon}(\lambda)$ and $O_{2,\varepsilon}(\lambda)$ on $\hat{\mathcal{M}}_\varepsilon$ in the resonance region for λ near $\bar{\lambda}$ and all small enough ε . The curve $O_{1,\varepsilon}(\lambda)$ can be either a stable periodic orbit for the restricted system (3.2) on $\hat{\mathcal{M}}_\varepsilon$ or a (restricted) unstable manifold of a saddle for this system. The curve $O_{2,\varepsilon}(\lambda)$ can be either an unstable periodic orbit for the restricted system (3.2) on $\hat{\mathcal{M}}_\varepsilon$ or a (restricted) stable manifold of a saddle for this system.

To set up the geometry for the first theorem, shown in figure 4, let for $\lambda = \bar{\lambda}$ the line

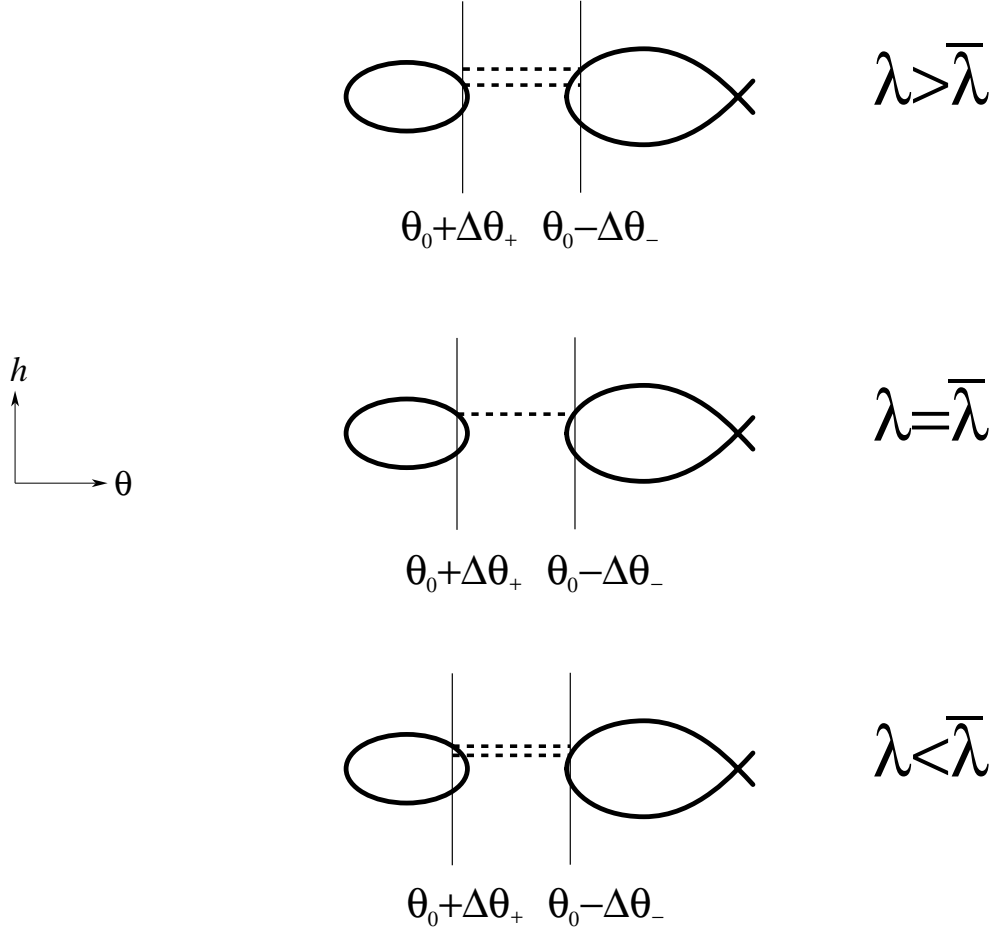


Figure 5: The transversality condition needed for theorem 1 to be valid.

$\theta = \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})$ intersect transversely the curve $O_{1,0}(\bar{\lambda})$, and the line $\theta = \bar{\theta}_0 + \Delta\theta_+(\bar{\phi})$ intersect transversely the curve $O_{2,0}(\bar{\lambda})$, and let both intersections occur at the same value of h . In this case a heteroclinic orbit on the limiting homoclinic surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ connects the two intersection points. Assume further that, for $\lambda > \bar{\lambda}$, the h -coordinate of the intersection of the line $\theta = \bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))$ and the curve $O_{1,0}(\lambda)$ is larger (smaller) than the h -coordinate of the intersection of the line $\theta = \bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda))$ and the curve $O_{2,0}(\lambda)$, and that for $\lambda < \bar{\lambda}$, the h -coordinate of the intersection of the line $\theta = \bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))$ and the curve $O_{1,0}(\lambda)$ is smaller (larger) than the h -coordinate of the intersection of the line $\theta = \bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda))$ and the curve $O_{2,0}(\lambda)$; see figure 5. In other words, the difference of these h -coordinates passes through zero transversely as λ passes through $\bar{\lambda}$. We will then show that for some $\lambda = \lambda(\varepsilon)$, there exists a heteroclinic connection between the orbits $O_{1,\varepsilon}(\lambda(\varepsilon))$ and $O_{2,\varepsilon}(\lambda(\varepsilon))$, for all small enough ε .

We now proceed to formalize this discussion. We begin by denoting the h -coordinates

of the two intersections discussed in the previous paragraph by $h\left(\bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))\right)$ and $h\left(\bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda))\right)$, respectively. Rather than to calculate the difference of these two h -coordinates, it is more convenient to calculate their squares, using equation (3.4). Thus,

$$\frac{1}{2}h^2\left(\bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))\right) = \mathcal{H}(O_{1,0}(\lambda)) - V\left(\bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda)), \lambda\right),$$

and

$$\frac{1}{2}h^2\left(\bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda))\right) = \mathcal{H}(O_{2,0}(\lambda)) - V\left(\bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda)), \lambda\right),$$

where $\mathcal{H}(O_{1,0}(\lambda))$ and $\mathcal{H}(O_{2,0}(\lambda))$ are the respective values of the rescaled Hamiltonian (3.4) on the curves $O_{1,0}(\lambda)$ and $O_{2,0}(\lambda)$. Then, we have

Theorem 1 *Let the curve $O_{1,\varepsilon}(\lambda)$ be either a stable periodic orbit for the restricted system (3.2) on $\hat{\mathcal{M}}_\varepsilon$ or a (restricted) unstable manifold of a saddle $s_{1,\varepsilon}(\lambda)$ for this system, and let the curve $O_{2,\varepsilon}(\lambda)$ be either an unstable periodic orbit for the restricted system (3.2) on $\hat{\mathcal{M}}_\varepsilon$ or a (restricted) stable manifold of a saddle $s_{2,\varepsilon}(\lambda)$ for this system. Moreover, let for $\lambda = \bar{\lambda}$, the line $\theta = \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})$ intersect transversely the curve $O_{1,0}(\bar{\lambda})$, and the line $\theta = \bar{\theta}_0 + \Delta\theta_+(\bar{\phi})$ intersect transversely the curve $O_{2,0}(\bar{\lambda})$. Finally, let*

$$\mathcal{H}(O_{1,0}(\bar{\lambda})) - V\left(\bar{\theta}_0 - \Delta\theta_-(\bar{\phi}), \bar{\lambda}\right) - \left[\mathcal{H}(O_{2,0}(\bar{\lambda})) - V\left(\bar{\theta}_0 + \Delta\theta_+(\bar{\phi}), \bar{\lambda}\right)\right] = 0, \quad (3.7)$$

and

$$\begin{aligned} \frac{d}{d\lambda} \left\{ \mathcal{H}(O_{1,0}(\lambda)) - V\left(\bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda)), \lambda\right) \right. \\ \left. - \left[\mathcal{H}(O_{2,0}(\lambda)) - V\left(\bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda)), \lambda\right) \right] \right\} \neq 0. \end{aligned} \quad (3.8)$$

Then, for all small enough ε and for some $\lambda = \lambda(\varepsilon)$ with $\lambda(0) = \bar{\lambda}$, there exists a heteroclinic orbit connecting either the periodic orbit $O_{1,\varepsilon}(\lambda(\varepsilon))$ or the saddle $s_{1,\varepsilon}(\lambda)$ to either the periodic orbit $O_{2,\varepsilon}(\lambda(\varepsilon))$ or the saddle $s_{2,\varepsilon}(\lambda)$.

We remark that a saddle of the system (3.2) must always exist near a saddle of the outer system (3.3) by proposition 5.3, and that a sufficient condition for the existence of limit cycles in the phase plane of the system (3.2) and a criterion for their stability is given in proposition 5.5.

We next turn our attention to the situation that involves a heteroclinic connection between a saddle or a stable limit cycle on the annulus $\hat{\mathcal{M}}_\varepsilon$ and a sink or another stable limit cycle on $\hat{\mathcal{M}}_\varepsilon$, such as the one shown in figure 6. This situation is described in

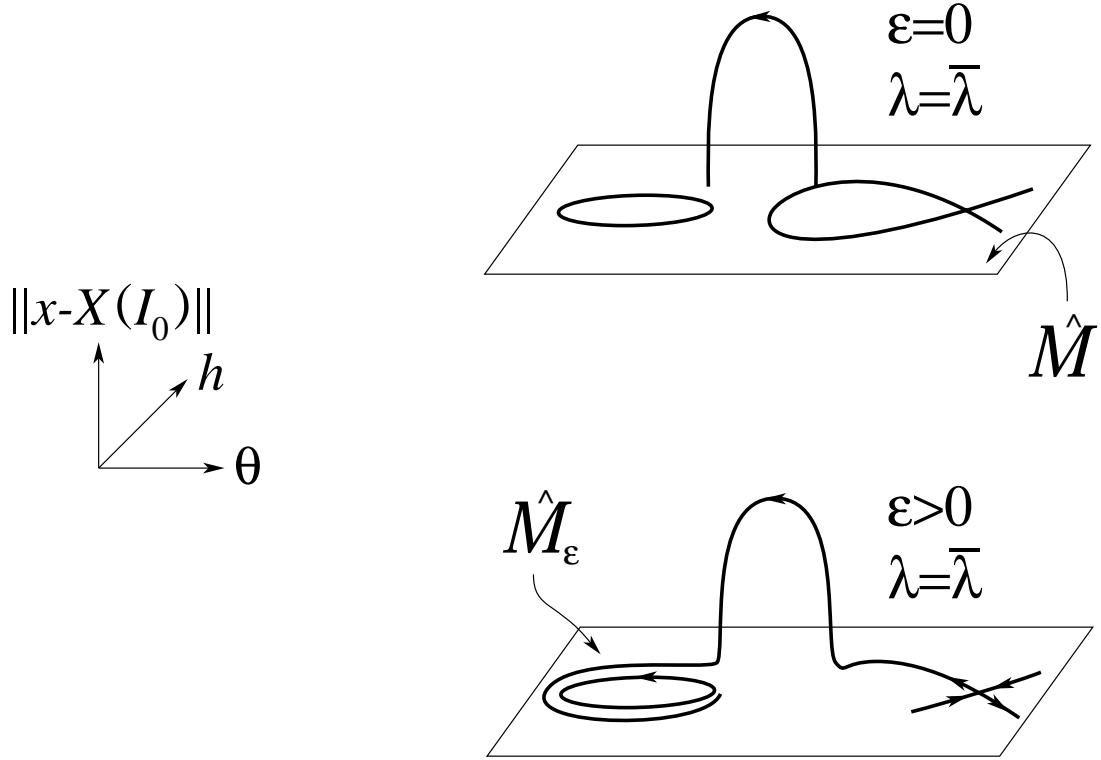


Figure 6: The situation discussed in theorem 2 where a heteroclinic orbit connects the saddle $s_\varepsilon(\lambda(\varepsilon))$ to the periodic orbit $O_\varepsilon(\lambda(\varepsilon))$, which is a stable limit cycle on the annulus $\hat{\mathcal{M}}_\varepsilon$.

Theorem 2 *Let the curve $O_{1,\varepsilon}(\lambda)$ be either a stable periodic orbit on the annulus $\hat{\mathcal{M}}_\varepsilon$, or the (restricted) unstable manifold of a saddle $s_\varepsilon(\lambda)$ on the annulus $\hat{\mathcal{M}}_\varepsilon$ for all λ near $\lambda = \bar{\lambda}$, and all small enough positive ε . At $\lambda = \bar{\lambda}$, let the curve $O_{1,0}(\bar{\lambda})$ and the line $\theta = \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})$ intersect transversely at some height $h = \bar{h}$. Furthermore, let the point $(h, \theta) = (\bar{h}, \bar{\theta}_0 + \Delta\theta_+(\bar{\phi}))$ lie in a compact domain \mathcal{R} that is all contained in the open region \mathcal{B} , the limit as $\varepsilon \rightarrow 0$ of the basin of attraction \mathcal{B}_ε of either an equilibrium, $c_\varepsilon(\lambda)$, which is a sink for the restricted system (3.2) on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$, or a periodic orbit, $O_{2,\varepsilon}(\lambda)$, which is a stable limit cycle for the restricted system (3.2) on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$. Then, for all small enough positive ε and all λ close enough to $\lambda = \bar{\lambda}$, there exists a heteroclinic orbit connecting either the periodic orbit $O_{1,\varepsilon}(\lambda)$ or the saddle $s_\varepsilon(\lambda)$ to either the equilibrium $c_\varepsilon(\lambda)$ or the periodic orbit $O_{2,\varepsilon}(\lambda)$. Moreover, the intersection of the unstable manifolds $W^u(O_{1,\varepsilon}(\lambda))$ or $W^u(s_\varepsilon(\lambda))$ with the stable manifolds $W^s(c_\varepsilon(\lambda))$ or $W^s(O_{2,\varepsilon}(\lambda))$ is transverse along that heteroclinic orbit.*

Sufficient conditions for the existence of sources, sinks, and limit cycles in the phase plane of system (3.2) and criteria for their stability are given in propositions 5.3 and 5.5, respectively. The existence of the limiting region \mathcal{B} is a part of the assumption, and needs to be checked

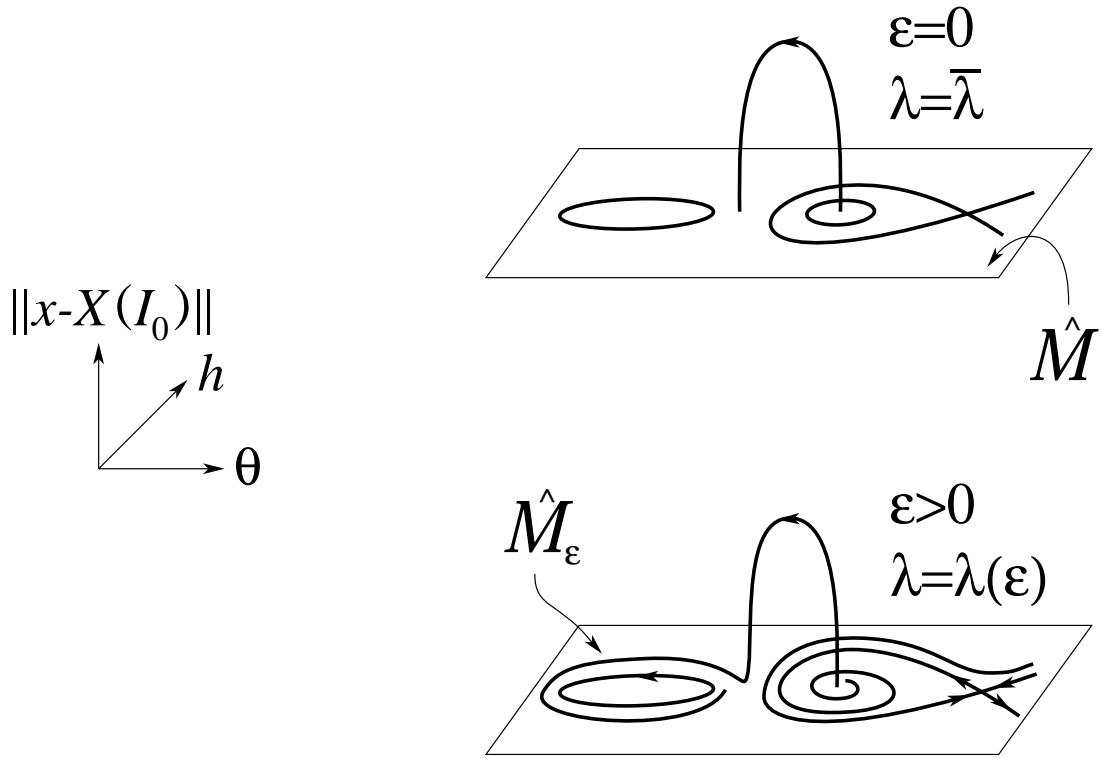


Figure 7: The situation discussed in theorem 3 where a heteroclinic orbit connects the spiral-saddle $c_\varepsilon(\lambda(\varepsilon))$ to the periodic orbit $O_\varepsilon(\lambda(\varepsilon))$. This periodic orbit is a stable limit cycle on the annulus \mathcal{M}_ε .

in each practical case separately. Also, by inverting the time, we can use this theorem to show the existence of a heteroclinic connection between a saddle or an unstable limit cycle on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$ and a source or another unstable limit cycle on that annulus.

Finally, we discuss the situation which involves a heteroclinic connection between two equilibria that are sinks for the restricted system (3.2), or a sink and a stable periodic orbit for that system; see figure 7. This situation is described in

Theorem 3 *Let $c_0(\lambda)$ be a center for the outer system (3.3), and let it be, at $\lambda = \bar{\lambda}$, located at*

$$(h(c_0(\bar{\lambda})), \theta(c_0(\bar{\lambda}))) = (0, \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})), \quad (3.9)$$

with

$$\frac{d}{d\lambda} [\theta(c_0(\lambda)) - \bar{\theta}_0(I_0, \lambda) + \Delta\theta_-(\bar{\phi}(I_0, \lambda))] \neq 0 \quad (3.10)$$

at $\lambda = \bar{\lambda}$. Let the corresponding perturbed equilibrium $c_\varepsilon(\lambda)$ be a sink for the restricted system (3.2) for all small enough ε and all λ close enough to $\lambda = \bar{\lambda}$. Finally, let the point $(h, \theta) = (0, \bar{\theta}_0 + \Delta\theta_+(\bar{\phi}))$ lie in a compact domain \mathcal{R} that is all contained in the open region

\mathcal{B} , the limit as $\varepsilon \rightarrow 0$ of the basin of attraction \mathcal{B}_ε of either an equilibrium, $d_\varepsilon(\lambda)$, which is a sink for the restricted system (3.2) on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$, or a periodic orbit, $O_\varepsilon(\lambda)$, which is a stable limit cycle for the restricted system (3.2) on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$. Then, for small $\varepsilon > 0$, there exists a function $\lambda = \lambda(\varepsilon)$ with $\lambda(0) = \bar{\lambda}$, such that there exists a heteroclinic orbit connecting the equilibrium $c_\varepsilon(\lambda(\varepsilon))$ to either the equilibrium $d_\varepsilon(\lambda(\varepsilon))$, or the periodic orbit $O_\varepsilon(\lambda(\varepsilon))$.

This theorem is a generalization of the result of ref. [30]. We remark that, again by inverting the time, we can use theorem 3 to show the existence of a heteroclinic connection between two equilibria that are sources for the restricted system (3.2), or a source and an unstable limit cycle for (3.2).

The rest of the paper is devoted to the necessary background and the proofs of theorems 1, 2 and 3.

4 Homoclinic Intersection Surfaces

In this section we first discuss the persistence of the annulus \mathcal{M} and its stable and unstable manifolds, $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$, for nonzero ε . We then review those features of the Melnikov method [1–17] that are necessary for understanding the rest of this paper. In particular, we focus our attention on calculating when the stable and unstable manifolds, $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$, of the perturbed annulus \mathcal{M}_ε intersect transversely, and what the nature of those intersections is. Alternatively, we can ask ourselves which homoclinic orbits will survive under perturbation.

In order to show persistence under perturbation of the annulus \mathcal{M} and its stable and unstable manifolds, we will have to perform local analysis around the annulus \mathcal{M} , and hence we now define the local stable and unstable manifold of \mathcal{M} . We pick a small positive δ and choose a neighborhood

$$U_\delta = \{(x, I, \theta) \mid I_1 < I < I_2, \|x - X(I)\| < \delta, 0 \leq \theta \leq 2\pi\}$$

of the annulus \mathcal{M} . We define the *local stable manifold*, $W_{loc}^s(\mathcal{M})$, of \mathcal{M} to be the component of $W^s(\mathcal{M}) \cap U_\delta$ whose points do not leave the neighborhood U_δ in forward time. Thus, the local stable manifold $W_{loc}^s(\mathcal{M})$ consists precisely of those points in the neighborhood U_δ which asymptote towards the annulus \mathcal{M} in positive time without ever leaving U_δ . We define $W_{loc}^u(\mathcal{M})$, the *local unstable manifold* of the annulus \mathcal{M} , in an analogous fashion. If κ is any

number smaller than $\inf\{\kappa(I) \mid I_1 < I < I_2\}$, where $\kappa(I)$ is the smallest positive real part of the eigenvalues of the matrix $JD_x^2 H(X(I), I)$, then trajectories on the local stable manifold $W_{loc}^s(\mathcal{M})$ approach the annulus \mathcal{M} in forward time exponentially at least at the rate $e^{-\kappa t}$. A similar statement is true for trajectories on the local unstable manifold $W_{loc}^u(\mathcal{M})$ in backward time.

We are now ready to state the precise result that describes how the annulus \mathcal{M} and its local stable and unstable manifolds persist under perturbation:

Proposition 4.1 *For all small enough positive ε , there exist a two-dimensional, locally invariant annular surface, \mathcal{M}_ε , and $(n+2)$ -dimensional, locally invariant manifolds, $W_{loc}^s(\mathcal{M}_\varepsilon)$ and $W_{loc}^u(\mathcal{M}_\varepsilon)$, inside the neighborhood U_δ , possessing the following properties:*

1. *The annulus \mathcal{M}_ε and the manifolds $W_{loc}^s(\mathcal{M}_\varepsilon)$ and $W_{loc}^u(\mathcal{M}_\varepsilon)$ vary smoothly with ε and other parameters in the problem.*
2. *For $\varepsilon = 0$, the annulus \mathcal{M}_ε and the manifolds $W_{loc}^s(\mathcal{M}_\varepsilon)$ and $W_{loc}^u(\mathcal{M}_\varepsilon)$ coincide with the annulus \mathcal{M} and the manifolds $W_{loc}^s(\mathcal{M})$ and $W_{loc}^u(\mathcal{M})$, respectively. For nonzero ε , the annulus \mathcal{M}_ε and the manifolds $W_{loc}^s(\mathcal{M}_\varepsilon)$ and $W_{loc}^u(\mathcal{M}_\varepsilon)$ can be written as smooth graphs over their unperturbed counterparts. In particular, the annulus \mathcal{M}_ε is given by the equation $x = X_\varepsilon(I, \theta, \lambda, \varepsilon)$ for some smooth function $X_\varepsilon(I, \theta, \lambda, \varepsilon)$ with $X_0(I, \theta, \lambda, 0) = X(I)$.*
3. *The manifolds $W_{loc}^s(\mathcal{M}_\varepsilon)$ and $W_{loc}^u(\mathcal{M}_\varepsilon)$ intersect along the annulus \mathcal{M}_ε .*
4. *Let κ be any number smaller than $\inf\{\kappa(I) \mid I_1 < I < I_2\}$, where $\kappa(I)$ is the smallest positive real part of the eigenvalues of the matrix $JD_x^2 H(X(I), I)$. Then any trajectory that starts at $t = 0$ inside the manifold $W_{loc}^s(\mathcal{M}_\varepsilon)$ will approach the annulus \mathcal{M}_ε in forward time at an exponential rate at least as fast as $e^{-\kappa t}$, as long as it stays in the manifold $W_{loc}^s(\mathcal{M}_\varepsilon)$. Any trajectory that starts at $t = 0$ inside the manifold $W_{loc}^u(\mathcal{M}_\varepsilon)$ will approach the annulus \mathcal{M}_ε in backward time at an exponential rate at least as fast as $e^{\kappa t}$, as long as it stays in the manifold $W_{loc}^u(\mathcal{M}_\varepsilon)$.*

PROOF: The proof of this theorem follows from refs. [44–46]. Details are similar to those in [17, p. 354]. \square

We call $W_{loc}^s(\mathcal{M}_\varepsilon)$ and $W_{loc}^u(\mathcal{M}_\varepsilon)$ the *local stable and unstable manifolds* of the perturbed annulus \mathcal{M}_ε , respectively. The meaning of the statement that the annulus \mathcal{M}_ε and its local

stable and unstable manifolds $W_{loc}^s(\mathcal{M}_\varepsilon)$ and $W_{loc}^u(\mathcal{M}_\varepsilon)$ are locally invariant is that they are spanned by orbits, but points can enter or leave them through their boundaries.

We define the *stable manifold*, $W^s(\mathcal{M}_\varepsilon)$, of the perturbed annulus \mathcal{M}_ε as the manifold obtained by evolving points on the local stable manifold $W_{loc}^s(\mathcal{M}_\varepsilon)$ in backward time. We note that trajectories can leave (but not enter) the stable manifold $W^s(\mathcal{M}_\varepsilon)$ through its boundary, which is enough to make the manifold $W^s(\mathcal{M}_\varepsilon)$ only locally invariant. This fact is in contrast with the usual properties of the stable manifold of an invariant manifold, which is itself invariant, and comes about because the perturbed annulus \mathcal{M}_ε itself is only locally invariant. An analogous definition and comments hold for the *unstable manifold*, $W^u(\mathcal{M}_\varepsilon)$, of the perturbed annulus \mathcal{M}_ε .

Gronwall-type estimates yield

Proposition 4.2 *Two trajectories, one on the unperturbed stable manifold $W^s(\mathcal{M})$ and the other on the perturbed stable manifold $W^s(\mathcal{M}_\varepsilon)$, which start a distance $\mathcal{O}(\varepsilon)$ apart at $t = 0$, will stay $\mathcal{O}(\varepsilon)$ close for all finite times. The same statement also holds for pairs of trajectories on the unperturbed unstable manifold $W^u(\mathcal{M})$ and the perturbed unstable manifold $W^u(\mathcal{M}_\varepsilon)$.*

We now turn our attention to the question of which unperturbed homoclinic orbit will survive under perturbation. As mentioned in section 3, the answer to this question is determined by the transverse zeros of the Melnikov vector (2.6). In particular, the equation $\mathbf{M}(I, \phi, \theta_0, \lambda) = 0$ presents n constraints on $(n + 2)$ variables, I , ϕ , θ_0 , and λ . Hence, if we fix the parameter λ , we expect this equation to provide a one-parameter family of surviving homoclinic orbits, or, in other words, a two-dimensional homoclinic intersection surface. This discussion is made precise in

Proposition 4.3 *Let for $I = \bar{I}$, $\phi = \bar{\phi}$, $\theta_0 = \bar{\theta}_0$ and $\lambda = \bar{\lambda}$ the following two statements be true:*

1. $\mathbf{M}(\bar{I}, \bar{\phi}, \bar{\theta}_0, \bar{\lambda}) = 0$.
2. *The matrix $D_{(\phi, \theta_0)}\mathbf{M}(\bar{I}, \bar{\phi}, \bar{\theta}_0, \bar{\lambda})$ is nonsingular.*

Then for ε sufficiently small, all I sufficiently close to \bar{I} , and all λ sufficiently close to $\bar{\lambda}$, there exists a two-dimensional intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ of the manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ in the $x - I - \theta$ phase space. The intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ varies smoothly

with ε and λ . In the limit as $\varepsilon \rightarrow 0$, it tends to the surface $\Sigma^\lambda(\bar{\phi}, \bar{\theta}_0)$ that consists of the unperturbed homoclinic orbits parametrized by t and I , with $\phi = \bar{\phi}(I, \lambda)$, and $\theta_0 = \bar{\theta}_0(I, \lambda)$, for I close enough to \bar{I} at the given λ . The manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ intersect transversely at every point of the intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$.

By assumption 5, the hypotheses of this theorem are satisfied for $I = I_0$ and some $\phi = \bar{\phi}$, $\theta_0 = \bar{\theta}_0$ and $\lambda = \bar{\lambda}$.

PROOF OF PROPOSITION 4.3: Let $\mathbf{M}(\bar{I}, \bar{\phi}, \bar{\theta}_0, \bar{\lambda}) = 0$, and let the matrix of partial derivatives $D_{(\phi, \theta)} \mathbf{M}(\bar{I}, \bar{\phi}, \bar{\theta}_0, \bar{\lambda})$ be nonsingular. Then, by the implicit function theorem, there exist functions $\phi = \bar{\phi}(I, \lambda)$, $\theta_0 = \bar{\theta}_0(I, \lambda)$ with $\bar{\phi}(\bar{I}, \bar{\lambda}) = \bar{\phi}$ and $\bar{\theta}_0(\bar{I}, \bar{\lambda}) = \bar{\theta}_0$, such that $\mathbf{M}(I, \bar{\phi}(I, \lambda), \bar{\theta}_0(I, \lambda), \lambda) = 0$, and that $D_{(\phi, \theta)} \mathbf{M}(I, \bar{\phi}(I, \lambda), \bar{\theta}_0(I, \lambda), \lambda)$ has maximal rank in some small neighborhood of the point $(\bar{I}, \bar{\lambda})$ in the $I - \lambda$ space. The proof now follows from theorems 4.1.9 and 4.1.10 in [17]. \square

We now discuss the nature of the orbits contained in the intersections of the manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$. We remark that even though a trajectory on an intersection orbit is $\mathcal{O}(\varepsilon)$ close to the trajectory given by $(x^h(t, I, \bar{\phi}(I, \lambda)), I, \bar{\theta}^h(t, I, \bar{\phi}(I, \lambda)) + \bar{\theta}_0(I, \lambda))$ for all finite $t \in \mathbb{R}$, the two orbits traced by these two trajectories are not necessarily uniformly close. Namely, by proposition 4.2, the two trajectories are only guaranteed to be uniformly close on compact intervals of t . After that, they may move away from each other, since the usual Gronwall-type estimate (see, for instance, [33]) only bounds their distance as $\varepsilon C_1 e^{C_2 |t|}$ where C_1 and C_2 are some appropriate positive constants.

We also note that an intersection orbit is contained in the stable manifold $W^s(\mathcal{M}_\varepsilon)$ forever in backward time and in the unstable manifold $W^u(\mathcal{M}_\varepsilon)$ forever in forward time by the very construction of those manifolds. However, in forward time, an intersection orbit may leave the stable manifold $W^s(\mathcal{M}_\varepsilon)$ through its boundary because this manifold is only locally invariant. It can also leave the unstable manifold $W^u(\mathcal{M}_\varepsilon)$ through its boundary in backward time for the same reason. Therefore, we expect most intersection orbits not to asymptote to any invariant object in the annulus \mathcal{M}_ε in either forward or backward time. However, all the orbits that do asymptote to invariant objects in the perturbed annulus \mathcal{M}_ε in both forward and backward time must be contained in the intersection of its stable and unstable manifolds, $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$.

If $\bar{I} = I_0$, and if we set $I = I_0 + \sqrt{\varepsilon}h$ and let $\varepsilon \rightarrow 0$, it should be clear that the homoclinic intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ tends, in the inner limit, to the limiting homoclinic intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$, described in section 3.

5 The Resonance Band

In this section we present a way to analyze the dynamics in the resonance band that emerges from the breakup of the circle of equilibria at $I = I_0$ for nonzero ε . We have seen in proposition 4.1 that the x coordinates of points on the perturbed annulus \mathcal{M}_ε are given by the equality $x = X_\varepsilon(I, \theta, \lambda, \varepsilon)$. Therefore, the following two equations completely describe the dynamics on the annulus \mathcal{M}_ε :

$$\dot{I} = \varepsilon g^I(X_\varepsilon(I, \theta, \lambda, \varepsilon), I, \theta, \lambda), \quad (5.1a)$$

$$\dot{\theta} = \Omega(X_\varepsilon(I, \theta, \lambda, \varepsilon), I) + \varepsilon g^\theta(X_\varepsilon(I, \theta, \lambda, \varepsilon), I, \theta, \lambda). \quad (5.1b)$$

Since we are interested in the dynamics in the resonance band near $I = I_0$, we also substitute $I = I_0 + \sqrt{\varepsilon}h$ into equations (5.1). We remark that the scaling factor in front of h is $\sqrt{\varepsilon}$ because we have assumed in assumption 3 the generic situation in which $\Omega(X(I_0), I_0) = 0$ and $\frac{d\Omega}{dI}(X(I_0), I_0) \neq 0$. Otherwise, this factor may be different; see [32-34].

We can only extract useful information about equations (5.1) near the resonance at $I = I_0$ if we can explicitly calculate, or at least approximate, the function $X_\varepsilon(I_0 + \sqrt{\varepsilon}h, \theta, \lambda, \varepsilon)$. This is indeed the case, as we now show. First, as a consequence of proposition 4.1, the function $X_\varepsilon(I, \theta, \lambda, \varepsilon)$ is smooth, so that it can be Taylor expanded about $\varepsilon = 0$:

$$x = X(I) + \varepsilon X_1(I, \theta, \lambda) + \varepsilon^2 X_2(I, \theta, \lambda) + \cdots + \varepsilon^{m-1} X_{m-1}(I, \theta, \lambda) + \mathcal{O}(\varepsilon^m), \quad (5.2)$$

where m is at most equal to the number of continuous derivatives of the vector field (2.1). Second, at $I = I_0$, the terms $X_i(I_0, \theta, \lambda)$ in this Taylor expansion of the perturbed annulus \mathcal{M}_ε and their partial derivatives $D_I^j X_i(I_0, \theta, \lambda)$ may be calculated recursively, as is shown in

Proposition 5.1 *At the resonant value $I = I_0$, we have*

$$X_1(I_0, \theta, \lambda) = \left[JD_x^2 H(X(I_0), I_0) \right]^{-1} \left(g^I(X(I_0), I_0, \theta, \lambda) \frac{dX(I_0)}{dI} - g^x(X(I_0), I_0, \theta, \lambda) \right), \quad (5.3)$$

and

$$X_i(I_0, \theta, \lambda) = \left[JD_x^2 H(X(I_0), I_0) \right]^{-1} \Phi_i \left(X(I_0), \frac{dX(I_0)}{dI}, X_1(I_0, \theta, \lambda), D_I X_1(I_0, \theta, \lambda), \right. \\ \left. D_\theta X_1(I_0, \theta, \lambda), \dots, X_{i-1}(I_0, \theta, \lambda), D_I X_{i-1}(I_0, \theta, \lambda), D_\theta X_{i-1}(I_0, \theta, \lambda), I_0, \theta, \lambda \right), \quad (5.4)$$

for $i = 2, \dots, m-1$, and some smooth function Φ_i . Likewise, for $i = 1, \dots, m-1$ and $j = 1, \dots, m-1-i$, the derivative $D_I^j X_i(I_0, \theta, \lambda)$ can be computed in terms of the functions $X(I_0), \dots, d^{j+1}X(I_0)/dI^{j+1}, X_1(I_0, \theta, \lambda), \dots, D_I^{j+1}X_1(I_0, \theta, \lambda), D_\theta X_1(I_0, \theta, \lambda), \dots, D^j D_\theta X_1(I_0, \theta, \lambda), \dots, X_{i-1}(I_0, \theta, \lambda), \dots, D_I^{j+1}X_{i-1}(I_0, \theta, \lambda), D_\theta X_{i-1}(I_0, \theta, \lambda), \dots, D_I^j D_\theta X_{i-1}(I_0, \theta, \lambda), X_i(I_0, \theta, \lambda), \dots, D_I^{j-1}X_i(I_0, \theta, \lambda), D_\theta X_i(I_0, \theta, \lambda), \dots, D_I^{j-1}D_\theta X_i(I_0, \theta, \lambda)$.

PROOF: Let $X_\varepsilon = X_\varepsilon(I, \theta, \lambda, \varepsilon)$, and proceed as in [37] and [30]. By equation (2.1a), we have

$$\dot{X}_\varepsilon = JD_x H(X_\varepsilon, I) + \varepsilon g^x(X_\varepsilon, I, \theta, \lambda).$$

On the other hand, we obtain by the chain rule and equations (2.1b) and (2.1c) the equation

$$\begin{aligned} \dot{X}_\varepsilon &= D_I X_\varepsilon \dot{I} + D_\theta X_\varepsilon \dot{\theta} \\ &= D_I X_\varepsilon \varepsilon g^I(X_\varepsilon, I, \theta, \lambda) + D_\theta X_\varepsilon \left(\Omega(X_\varepsilon, I) + \varepsilon g^\theta(X_\varepsilon, I, \theta, \lambda) \right). \end{aligned}$$

We equate the two expressions for \dot{X}_ε , Taylor expand, and examine the $\mathcal{O}(\varepsilon^i)$ term for $i = 1, \dots, m-1$.

When $i = 1$, we obtain the equation

$$\begin{aligned} JD_x^2 H(X(I), I) X_1(I, \theta, \lambda) + g^x(X(I), I, \theta, \lambda) \\ = g^I(X(I), I, \theta, \lambda) \frac{dX(I)}{dI} + \Omega(X(I), I) D_\theta X_1(I, \theta, \lambda). \end{aligned}$$

Formula (5.3) now follows upon setting $I = I_0$ because $\Omega(X(I_0), I_0) = 0$ by assumption 3.

Similarly, for $i = 2, \dots, m-1$, we obtain the equation

$$\begin{aligned} JD_x^2 H(X(I), I) X_i(I, \theta, \lambda) \\ = \Phi_i \left(X(I), \frac{dX(I)}{dI}, X_1(I, \theta, \lambda), D_I X_1(I, \theta, \lambda), D_\theta X_1(I, \theta, \lambda), \dots, \right. \\ \left. X_{i-1}(I, \theta, \lambda), D_I X_{i-1}(I, \theta, \lambda), D_\theta X_{i-1}(I, \theta, \lambda), I, \theta, \lambda \right) \\ + \Omega(X(I), I) D_\theta X_i(I, \theta, \lambda). \end{aligned} \tag{5.5}$$

Upon setting $I = I_0$, formula (5.4) follows as above.

The expressions for the derivatives $D_\theta X_i(I_0, \theta, \lambda)$ can be computed by simply differentiating formulas (5.3) and (5.4) with respect to θ .

Finally, the statement about the derivatives $D_I^j X_i(I_0, \theta, \lambda)$ follows after differentiating formula (5.5), and setting $I = I_0$. \square

In practice, the first order corrections in the expansion about $\sqrt{\varepsilon} = 0$ of equations (5.1) with $I = I_0 + \sqrt{\varepsilon}h$ should suffice. Upon rescaling the time using $\tau = \sqrt{\varepsilon}t$ and setting $' = \frac{d}{d\tau}$, these equations read

$$h' = g^I(X(I_0), I_0, \theta, \lambda) + \sqrt{\varepsilon}F_h(h, \theta, \lambda) + \mathcal{O}(\varepsilon), \quad (5.6a)$$

$$\theta' = \frac{d\Omega}{dI}(X(I_0), I_0) h + \sqrt{\varepsilon}F_\theta(h, \theta, \lambda) + \mathcal{O}(\varepsilon), \quad (5.6b)$$

with

$$F_h(h, \theta, \lambda) = \frac{d}{dI} [g^I(X(I_0), I_0, \theta, \lambda)] h \quad (5.7)$$

and

$$F_\theta(h, \theta, \lambda) = \frac{1}{2} \frac{d^2\Omega(X(I_0), I_0)}{dI^2} h^2 + D_x\Omega(X(I_0), I_0)X_1(I_0, \theta, \lambda) + g^\theta(X(I_0), I_0, \theta, \lambda), \quad (5.8)$$

where $X_1(I_0, \theta, \lambda)$ is given by formula (5.3).

As stated in section 3, in the limit as $\varepsilon \rightarrow 0$, we obtain the outer system (3.3):

$$h' = g^I(X(I_0), I_0, \theta, \lambda), \quad \theta' = \frac{d\Omega}{dI}(X(I_0), I_0) h.$$

Since the outer system has the special form of a one-degree-of-freedom Newtonian system with the Hamiltonian function (3.4),

$$\mathcal{H}(h, \theta, \lambda) = \frac{1}{2} \frac{d\Omega}{dI}(X(I_0), I_0) h^2 + V(\theta, \lambda) = \frac{1}{2} \frac{d\Omega}{dI}(X(I_0), I_0) h^2 - \int_0^\theta g^I(X(I_0), I_0, s, \lambda) ds,$$

it is easy to analyze, and possesses certain simple general properties. (See ref. [47].) These properties are stated in

Proposition 5.2 *Orbits of the outer system (3.3) are symmetric about the θ -axis, and its equilibria can only lie on the θ -axis. Also, since the function $g^I(x, I, \theta, \lambda)$ is periodic in θ , the system(3.3) can only have an even number of equilibria for θ in $[0, 2\pi)$, provided that all the zeros of the expression $g^I(X(I_0), I_0, \theta, \lambda)$ are simple. Moreover, the derivative $D_\theta g^I(X(I_0), I_0, \theta, \lambda)$ must have opposite signs at two consecutive zeros θ_1 and θ_2 . Therefore, one of any two neighboring equilibria of (3.3) must be a center and the other a saddle.*

We notice that the potential part, $V(\theta, \lambda)$, of the Hamiltonian $\mathcal{H}(h, \theta, \lambda)$ can be written in the form

$$\begin{aligned} V(\theta, \lambda) &= \hat{V}(\theta, \lambda) - \bar{V}(\lambda)\theta \\ &= - \int_0^\theta [g^I(X(I_0), I_0, s, \lambda) - \bar{V}(\lambda)] ds - \bar{V}(\lambda)\theta \end{aligned} \quad (5.9)$$

where

$$\bar{V}(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} g^I(X(I_0), I_0, s, \lambda) ds.$$

The expression $\hat{V}(\theta, \lambda)$ in formula (5.9) is periodic in θ , and the expression $\bar{V}(\lambda)\theta$ is linear in θ . Formula (3.4) then shows that, say, if $\bar{V}(\lambda)$ and $\frac{d\Omega}{dt}(X(I_0), I_0)$ have the same sign, then every orbit on the $h - \theta$ phase cylinder must be bounded from the left. In particular, the left-hand halves of the stable and unstable manifolds of a saddle must either coincide along a homoclinic orbit or form two heteroclinic connections to another saddle. In both cases, the homoclinic orbit, or heteroclinic connections, encircle a center. A similar statement is true if $\bar{V}(\lambda)$ and $\frac{d\Omega}{dt}(X(I_0), I_0)$ have opposite signs.

We also notice that the decomposition (5.9) of the potential $V(\theta, \lambda)$ implies that the Hamiltonian $\mathcal{H}(h, \theta, \lambda)$ is not a single-valued function on the $h - \theta$ cylinder.

By assumption 4, the function $g^I(X(I_0), I_0, \theta, \lambda)$ has only a finite number of simple zeros; that is, the partial derivative $D_\theta g^I(X(I_0), I_0, \theta, \lambda)$ is nonzero at each zero of $g^I(X(I_0), I_0, \theta, \lambda)$. The implicit function theorem immediately implies

Proposition 5.3 *Every equilibrium of the system (3.3) persists in the system (3.2) a distance $\mathcal{O}(\sqrt{\varepsilon})$ away (in the $x - h - \theta$ coordinates), and also persists for neighboring values of the parameter λ . If the unperturbed equilibrium is a saddle, so is the perturbed one. A sufficient condition for an unperturbed center to perturb into a source or a sink is that the expression $D_h F_h(h, \theta, \lambda) + D_\theta F_\theta(h, \theta, \lambda)$ calculated at that center be positive or negative, respectively.*

The last sentence in this proposition is true because the real parts of both eigenvalues at the perturbed counterpart of a center are equal to

$$\frac{\sqrt{\varepsilon}}{2} [D_h F_h(h, \theta, \lambda) + D_\theta F_\theta(h, \theta, \lambda)] + \mathcal{O}(\varepsilon).$$

Therefore, the center perturbs into a source on the cylinder \mathcal{M}_ε if this expression is positive, and into a sink if this expression is negative. (Recall that the expressions $F_h(h, \theta, \lambda)$ and $F_\theta(h, \theta, \lambda)$ are given by formulas (5.7) and (5.8), respectively.)

The usual stable manifold theorem and Gronwall-type estimates imply

Proposition 5.4 *Let s_0 be a saddle for the outer system (3.3), and let s_ε be its perturbed counterpart for small positive ε . Then, a trajectory on the unstable manifold of the restricted system (3.2) of the perturbed saddle s_ε is $\mathcal{O}(\sqrt{\varepsilon})$ close (in the $x - h - \theta$ coordinates) to a trajectory on the unstable manifold of the unperturbed saddle s_0 on the τ -interval $(-\infty, T]$ for all finite T . A similar statement holds for the stable manifolds of the saddles s_ε and s_0 on τ -intervals $[T, \infty)$.*

Periodic orbits of the outer system (3.3) may also survive in the system (3.2):

Proposition 5.5 *A periodic orbit of the outer system (3.3) will survive in the system (3.2) if the subharmonic Melnikov function*

$$\begin{aligned} M(\mathcal{H}) &= \oint F_h(h, \theta, \lambda) d\theta - F_\theta(h, \theta, \lambda) dh \\ &= \oint \left[\frac{d\Omega}{dI}(X(I_0), I_0) h(\tau) F_h(h(\tau), \theta(\tau), \lambda) - g^I(X(I_0), I_0, \theta(\tau), \lambda) F_\theta(h(\tau), \theta(\tau), \lambda) \right] d\tau, \end{aligned}$$

calculated around that periodic orbit, has a transverse zero as a function of \mathcal{H} , the orbit's energy in the system (3.3). This orbit then also persists for neighboring λ . Moreover, if $dM(\mathcal{H})/d\mathcal{H}$ is positive on the persisting orbit, then this orbit is unstable on the perturbed annulus \mathcal{M}_ε ; and if $dM(\mathcal{H})/d\mathcal{H}$ is negative on the persisting orbit, then this orbit is stable.

For a special case, a proof of this proposition is given on p. 92 in ref. [47]. The more general case described here is proven in the same way.

6 Geometric Singular Perturbation Theory

In order to study the orbit structure on the stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ near the perturbed annulus \mathcal{M}_ε and near the resonance at $I = I_0$, we use the rescaling $I = I_0 + \sqrt{\varepsilon} h$ on the full system (2.1), to obtain the system

$$\dot{x} = JD_x H(x, I_0 + \sqrt{\varepsilon} h) + \varepsilon g^x(x, I_0 + \sqrt{\varepsilon} h, \theta, \lambda), \quad (6.1a)$$

$$\dot{h} = \sqrt{\varepsilon} g^I(x, I_0 + \sqrt{\varepsilon} h, \theta, \lambda), \quad (6.1b)$$

$$\dot{\theta} = \Omega(x, I_0 + \sqrt{\varepsilon} h) + \varepsilon g^\theta(x, I_0 + \sqrt{\varepsilon} h, \theta, \lambda). \quad (6.1c)$$

Setting $\varepsilon = 0$ in equations (6.1), we obtain the inner system (3.5):

$$\dot{x} = JD_x H(x, I_0), \quad \dot{h} = 0, \quad \dot{\theta} = \Omega(x, I_0).$$

The properties of its phase space are described in section 3. On the other hand, if we restrict the dynamics of equations (6.1) to the annulus \mathcal{M}_ε and rescale the time into $\tau = \sqrt{\varepsilon}t$, we obtain in the limit as $\varepsilon \rightarrow 0$ the outer system (3.3):

$$h' = g(X(I_0), I_0, \theta, \lambda), \quad \theta' = \frac{d\Omega(X(I_0), I_0)}{dI} h,$$

with $' = \frac{d}{d\tau}$. We studied this system in the previous section.

As we have already mentioned in section 3, the inner system (3.5) describes the dynamics away from the limiting annulus $\hat{\mathcal{M}}$ at $x = X(I_0)$, and the outer system (3.3) describes the rescaled slow dynamics on the annulus $\hat{\mathcal{M}}$. The work of Fenichel [37, 44–46] provides a means to connect the dynamics of systems (3.5) and (3.3) in order to achieve a description of the orbits in the local stable and unstable manifolds $W_{loc}^s(\hat{\mathcal{M}}_\varepsilon)$ and $W_{loc}^u(\hat{\mathcal{M}}_\varepsilon)$ of the annulus $\hat{\mathcal{M}}_\varepsilon$, the perturbed counterpart of the annulus $\hat{\mathcal{M}}$. First, proposition 4.1 applies to the system (6.1), and provides for smooth dependence of the annulus $\hat{\mathcal{M}}_\varepsilon$ and its stable and unstable manifolds $W_{loc}^s(\hat{\mathcal{M}}_\varepsilon)$ and $W_{loc}^u(\hat{\mathcal{M}}_\varepsilon)$ on ε up to including $\varepsilon = 0$. Second, theorem 9.1 in ref. [37] applies to systems of the same type as (6.1), and guarantees that the local manifolds $W_{loc}^s(\hat{\mathcal{M}}_\varepsilon)$ and $W_{loc}^u(\hat{\mathcal{M}}_\varepsilon)$ are foliated by *stable and unstable fibers*. The statement of this theorem, which is tailored to the needs of the present paper, is given in the next proposition. (See also ref. [40].) The estimates in this proposition are stated in the $x - h - \theta$ coordinates. The proposition is given for stable fibers; the proposition for unstable fibers is the same except for obvious changes.

Proposition 6.1 *For all small enough ε , the local stable manifold $W_{loc}^s(\hat{\mathcal{M}}_\varepsilon)$ of the invariant annulus $\hat{\mathcal{M}}_\varepsilon$ is foliated by a family of disjoint n -dimensional manifolds called stable fibers. These stable fibers have the following additional properties:*

1. *They form a locally positively invariant family; that is, the image (under the forward time flow) of any stable fiber is contained in a stable fiber as long as this image is contained in the local stable manifold $W_{loc}^s(\hat{\mathcal{M}}_\varepsilon)$.*
2. *Each stable fiber pierces the annulus $\hat{\mathcal{M}}_\varepsilon$ transversely inside the manifold $W_{loc}^s(\hat{\mathcal{M}}_\varepsilon)$ in precisely one point, called its base point.*

3. As the base points on the annulus $\hat{\mathcal{M}}_\varepsilon$ move under the dynamics of the vector field (6.1), the stable fibers move along with their base points and contract exponentially towards their base points in forward time as long as the base points stay in $\hat{\mathcal{M}}_\varepsilon$. The rate of this exponential contraction is at least $e^{-\kappa t}$, where κ is any number smaller than the smallest positive real part $\kappa(I_0)$ of all the eigenvalues of the equation (2.2a) at $I = I_0$ (or, equivalently, the inner equation (3.5a)) linearized around $x = X(I_0)$.
4. The family of stable fibers varies smoothly with $\sqrt{\varepsilon}$, λ , and any other parameters in the problem.
5. For $\varepsilon = 0$, that is, for system (3.5), the stable fibers are precisely the local stable manifolds of the equilibria on the $h - \theta$ annulus $\hat{\mathcal{M}}$.

Theorems similar to proposition 6.1 are also stated in refs. [30, 38, 48].

Fenichel's fibers make it possible to construct local stable and unstable manifolds of periodic orbits and equilibria on the annulus $\hat{\mathcal{M}}_\varepsilon$:

Proposition 6.2 *The local stable and unstable manifolds of objects on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$ are characterized by the following properties:*

1. For every stable periodic orbit on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$, its local unstable manifold is the union of all the unstable fibers whose base points lie on that orbit. The local part (contained in the neighborhood U_δ) of its stable manifold is the union of all the stable fibers whose base points are contained in the forward-time basin of attraction, \mathcal{B}_ε , of this periodic orbit on the annulus $\hat{\mathcal{M}}_\varepsilon$.
2. For every unstable periodic orbit on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$, its local stable manifold is the union of all the stable fibers whose base points lie on that orbit. The local part (contained in the neighborhood U_δ) of its unstable manifold is the union of all the unstable fibers whose base points are contained in the backward-time basin of attraction, \mathcal{B}_ε , of this periodic orbit on the annulus $\hat{\mathcal{M}}_\varepsilon$.
3. For every equilibrium in the resonance that is a sink for the system (6.1) restricted to the annulus $\hat{\mathcal{M}}_\varepsilon$, that is, equations (3.2), its local unstable manifold is precisely the unstable fiber having this sink as its base point. The local part (contained in the neighborhood U_δ) of its stable manifold is the union of all the stable fibers whose base

points are contained in the forward-time basin of attraction, \mathcal{B}_ε , of this sink on the annulus $\hat{\mathcal{M}}_\varepsilon$.

4. For every equilibrium in the resonance that is a source for the equations (3.2), its local stable manifold is precisely the stable fiber having this source as its base point. The local part (contained in the neighborhood U_δ) of its unstable manifold is the union of all the unstable fibers whose base points are contained in the backward-time basin of attraction, \mathcal{B}_ε , of this source on the annulus $\hat{\mathcal{M}}_\varepsilon$.
5. For every equilibrium that is a saddle for the restricted system (3.2), the local parts (that lie in the neighborhood U_δ) of its stable and unstable manifolds are the unions of the stable and unstable fibers with base points lying on the restricted stable and unstable manifolds of this saddle on the annulus $\hat{\mathcal{M}}_\varepsilon$.

This proposition is similar to theorems 12.1, 12.2, 13.1, and 13.2 in ref. [37].

We obtain the full global stable and unstable manifolds of orbits and equilibria that lie on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$ by evolving their local counterparts in forward and backward time, respectively.

Using both systems of equations, (3.3) and (3.5), we can obtain the limiting structure of selected objects on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$ and their stable and unstable manifolds. In particular, let O_ε be an orbit on $\hat{\mathcal{M}}_\varepsilon$ that limits, as $\varepsilon \rightarrow 0$, onto a curve O_0 . This curve is a level curve of the rescaled Hamiltonian $\mathcal{H}(h, \theta)$ at some value $\mathcal{H}(h, \theta) = \mathcal{H}_0$. The curve O_0 is an orbit for the outer equations (3.3), and a curve of equilibria for the inner equations (3.5).

If O_ε is a periodic orbit, it is clear what is meant by its local stable and unstable manifolds. However, let O_ε be a piece of the unstable manifold of a saddle for the restricted system (3.2) on the annulus $\hat{\mathcal{M}}_\varepsilon$, parametrized by a trajectory on a τ -interval $(-\infty, T]$ for some large positive T . In this case, we *define* the local unstable manifold $W_{loc}^u(O_\varepsilon)$ to be the union of all the unstable fibers whose base points lie on the curve O_ε . By proposition 5.4, there exists a piece of the stable manifold of a saddle for the limiting outer system 3.3 on $\hat{\mathcal{M}}$ that is also parametrized by a trajectory on the τ -interval $(-\infty, T]$ and is $\mathcal{O}(\sqrt{\varepsilon})$ close to O_ε . This piece is the limiting curve O_0 . An analogous definition can be given for the stable manifold $W_{loc}^s(O_\varepsilon)$ if the orbit O_ε is a piece of the stable manifold of a saddle for the restricted system 3.2.

The local stable and unstable manifolds, $W_{loc}^s(O_0)$ and $W_{loc}^u(O_0)$, of the limiting curve O_0 are the unions of the n -dimensional stable and unstable manifolds of the equilibria (under

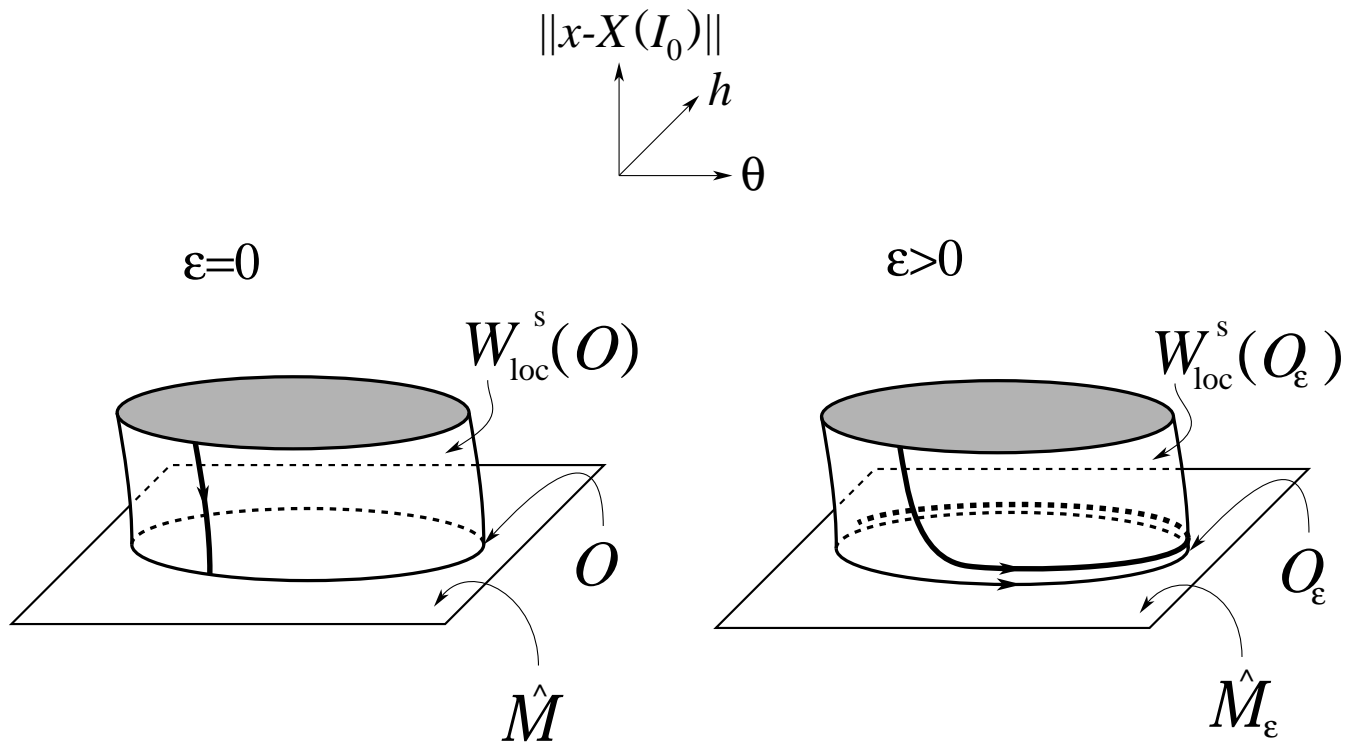


Figure 8: As $\varepsilon \rightarrow 0$ the local unstable manifold $W_{loc}^u(O_\varepsilon)$ of the periodic orbit O_ε limits onto the local unstable manifold $W_{loc}^u(O_0)$ of the limiting curve O_0 . This curve is a periodic orbit of the outer system

the dynamics of the inner equations (3.5)) comprising the curve O_0 , respectively. The global stable and unstable manifolds of all the above-mentioned objects can now be defined in the usual way by evolving trajectories on the local stable and unstable manifolds in backward and forward time, respectively.

Propositions 6.1 and 6.2 now imply (in the $x-h-\theta$ coordinates) the following proposition, whose contents are illustrated in figures 8 and 9:

Proposition 6.3 *In the limit when $\varepsilon \rightarrow 0$, the following limiting structures can be constructed with the aid of stable and unstable fibers:*

1. *If O_ε is a stable periodic orbit on \hat{M}_ε for the restricted system (3.2), then its local unstable manifold, $W_{loc}^u(O_\varepsilon)$, limits, as $\varepsilon \rightarrow 0$, onto the local unstable manifold, $W_{loc}^u(O_0)$, of the limiting closed curve O_0 .*

2. If O_ε is an unstable periodic orbit on $\hat{\mathcal{M}}_\varepsilon$ for the restricted system (3.2), then its local stable manifold, $W_{loc}^s(O_\varepsilon)$, limits, as $\varepsilon \rightarrow 0$, onto the local stable manifold, $W_{loc}^s(O_0)$, of the limiting closed curve O_0 .
3. If O_ε is a piece of the restricted stable manifold of a saddle on $\hat{\mathcal{M}}_\varepsilon$ as discussed above, then its local stable manifold, $W_{loc}^s(O_\varepsilon)$, limits, as $\varepsilon \rightarrow 0$, onto the local stable manifold, $W_{loc}^s(O_0)$, of the limiting curve O_0 .
4. If O_ε is a piece of the restricted unstable manifold of a saddle on $\hat{\mathcal{M}}_\varepsilon$ as discussed above, then its local unstable manifold, $W_{loc}^u(O_\varepsilon)$, limits, as $\varepsilon \rightarrow 0$, onto the local unstable manifold, $W_{loc}^u(O_0)$, of the limiting curve O_0 .
5. If c_ε is a sink on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$ for the restricted system (3.2), then, as $\varepsilon \rightarrow 0$, its local unstable manifold limits onto the local unstable manifold of the limiting center, c_0 , on the annulus $\hat{\mathcal{M}}$.
6. If c_ε is a source on the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$ for the restricted system (3.2), then, as $\varepsilon \rightarrow 0$, its local stable manifold limits onto the local stable manifold of the limiting center, c_0 , on the annulus $\hat{\mathcal{M}}$.

In all of the above cases, the local stable and unstable manifolds of the curves O_ε and O_0 and the equilibria c_ε and c_0 are $\mathcal{O}(\sqrt{\varepsilon})$ apart, respectively.

7 Proofs of the Main Theorems

In this section we finally couple the dynamics near the resonance band in the annulus \mathcal{M}_ε with the dynamics on the surviving homoclinic orbits using the geometric singular perturbation theory discussed in the previous section.

We recall the limiting homoclinic intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$, which consists of those heteroclinic orbits connecting equilibria on the $h - \theta$ cylinder $\hat{\mathcal{M}}$ that emerge from the $h - \theta$ cylinder $\hat{\mathcal{M}}$ along the line $\theta = \bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))$ and return to $\hat{\mathcal{M}}$ along the line $\theta = \bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda))$. Here, by formulas (3.6),

$$\Delta\theta_+(\phi) = \int_0^\infty \Omega(x^h(s, I_0, \phi), I_0) ds, \quad \Delta\theta_-(\phi) = \int_{-\infty}^0 \Omega(x^h(s, I_0, \phi), I_0) ds.$$

We now proceed to prove theorem 1. In order to do this, we begin by proving two auxiliary propositions. The first proposition is a local transversality result:

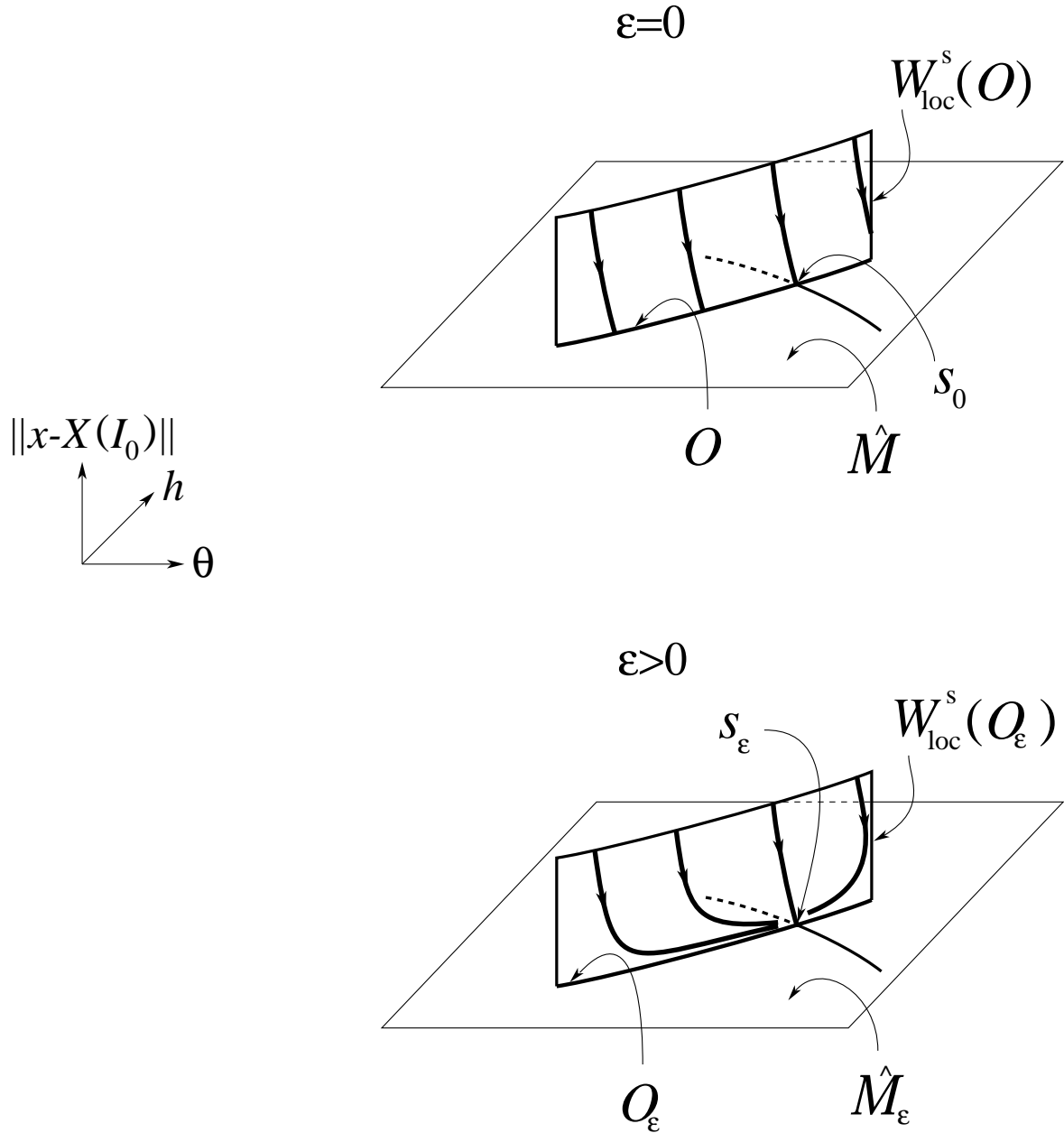


Figure 9: When the orbit segment O_ε is the restricted unstable manifold of a saddle s_ε on the annulus \hat{M}_ε , then its local unstable manifold $W_{loc}^u(O_\varepsilon)$ is a part of the unstable manifold of the saddle s_ε that is all contained in the δ -neighborhood U_δ of the annulus \hat{M} . As $\varepsilon \rightarrow 0$ the local unstable manifold $W_{loc}^u(O_\varepsilon)$ limits onto the local unstable manifold $W_{loc}^u(O_0)$ of the limiting curve O_0 . This curve is a segment of the unstable manifold of the limiting saddle s_0 for the outer system

Proposition 7.1 *Let $\varepsilon = 0$ and let for $\lambda = \bar{\lambda}$ the line $\theta = \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})$ intersect transversely the curve $O_{1,0}(\bar{\lambda})$, and the line $\theta = \bar{\theta}_0 + \Delta\theta_+(\bar{\phi})$ intersect transversely the curve $O_{2,0}(\bar{\lambda})$. Then for all λ near $\bar{\lambda}$, the local unstable manifold $W_{loc}^u(O_{1,0}(\lambda))$ of the curve $O_{1,0}(\lambda)$ intersects the limiting homoclinic intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ transversely inside $W_{loc}^u(\hat{\mathcal{M}})$, and the local stable manifold $W_{loc}^s(O_{2,0}(\lambda))$ of the curve $O_{2,0}(\lambda)$ intersects the limiting homoclinic intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ transversely inside $W_{loc}^s(\hat{\mathcal{M}})$.*

PROOF: We prove the first part of the proposition; the proof of the second part is almost identical. Recall that the manifold $W_{loc}^u(\hat{\mathcal{M}})$ is parametrized by t, h, ϕ and θ_0 in the expression $(x^h(t, I_0, \phi), h, \theta^h(t, I_0, \phi) + \theta_0)$. The tangent space at any point of $W_{loc}^u(\hat{\mathcal{M}})$ is therefore spanned by the vectors

$$\begin{aligned} & \left(\dot{x}^h(t, I_0, \phi), 0, \dot{\theta}^h(t, I_0, \phi) \right), \\ & \left(D_\phi x^h(t, I_0, \phi), 0, D_\phi \theta^h(t, I_0, \phi) \right), \\ & (0, 1, 0), \\ & (0, 0, 1). \end{aligned}$$

Now, since the curve $O_{1,0}(\bar{\lambda})$ intersects the vertical line $\theta = \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})$ transversely on the annulus $\hat{\mathcal{M}}$, the same must be true for the intersection of the curve $O_{1,0}(\lambda)$ and the vertical line $\theta = \bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))$ for all λ close enough to $\lambda = \bar{\lambda}$. Therefore, the curve $O_{1,0}(\lambda)$ must be expressible as a graph $h = h(\theta)$ near $\theta = \bar{\theta}_0(I_0, \lambda) - \Delta\theta_-(\bar{\phi}(I_0, \lambda))$. Thus the manifold $W_{loc}^u(O_{1,0}(\lambda))$ can be parametrized by t, ϕ and θ_0 in the expression $(x^h(t, I_0, \phi), h(\theta^h(-\infty, I_0, \phi) + \theta_0), \theta^h(t, I_0, \phi) + \theta_0)$. The tangent space at any point of the manifold $W_{loc}^u(O_{1,0}(\bar{\lambda}))$ is therefore spanned by the vectors

$$\begin{aligned} & \left(\dot{x}^h(t, I_0, \phi), 0, \dot{\theta}^h(t, I_0, \phi) \right), \\ & \left(D_\phi x^h(t, I_0, \phi), \frac{dh}{d\theta}(\theta^h(-\infty, I_0, \phi) + \theta_0) D_\phi \theta^h(-\infty, I_0, \phi), D_\phi \theta^h(t, I_0, \phi) \right), \\ & \left(0, \frac{dh}{d\theta}(\theta^h(-\infty, I_0, \phi) + \theta_0), 1 \right). \end{aligned}$$

Finally, the tangent space at any point of the limiting intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ is spanned by the two vectors

$$\begin{aligned} & \left(\dot{x}^h(t, I_0, \phi), 0, \dot{\theta}^h(t, I_0, \phi) \right), \\ & (0, 1, 0). \end{aligned}$$

It is therefore easy to see that the tangent spaces of the manifold $W_{loc}^u(O_{1,0}(\lambda))$ and the limiting intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ add up to the tangent space of the local unstable manifold $W_{loc}^u(\hat{\mathcal{M}})$. \square

The second auxiliary proposition uses the first proposition to prove the existence of two special orbits:

Proposition 7.2 *Let the curve $O_{1,\varepsilon}(\lambda)$ be either a stable periodic orbit for the restricted system (3.2) on $\hat{\mathcal{M}}_\varepsilon$ or a (restricted) unstable manifold of a saddle for this system. Let the curve $O_{2,\varepsilon}(\lambda)$ be either an unstable periodic orbit for the restricted system (3.2) on $\hat{\mathcal{M}}_\varepsilon$ or a (restricted) stable manifold of a saddle for this system. Moreover, let for $\lambda = \bar{\lambda}$ the line $\theta = \bar{\theta}_0 - \Delta\theta_-(\bar{\phi})$ intersect transversely the curve $O_{1,0}(\bar{\lambda})$, and the line $\theta = \bar{\theta}_0 + \Delta\theta_+(\bar{\phi})$ intersect transversely the curve $O_{2,0}(\bar{\lambda})$. Then for all λ near $\lambda = \bar{\lambda}$, and all small enough positive ε , there exists an orbit $a_{1,\varepsilon}^\lambda(t)$ that is contained in both the unstable manifold $W^u(O_{1,\varepsilon}(\lambda))$ and the intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$. Likewise, there exists an orbit $a_{2,\varepsilon}^\lambda(t)$ that is contained in both the stable manifold $W^s(O_{2,\varepsilon}(\lambda))$ and the intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$. Trajectories on both these orbits are $\mathcal{O}(\sqrt{\varepsilon})$ close (in the $x - h - \theta$ coordinates) to trajectories on the unperturbed counterparts of these orbits for all finite times t .*

Recall the definition of the local unstable manifolds of the curves $O_{1,\varepsilon}(\lambda(\varepsilon))$ and $O_{2,\varepsilon}(\lambda(\varepsilon))$ from section 6 in the case when $O_{1,\varepsilon}(\lambda(\varepsilon))$ is the (restricted) unstable manifold of a saddle on the annulus $\hat{\mathcal{M}}_\varepsilon$, or when $O_{2,\varepsilon}(\lambda(\varepsilon))$ is the (restricted) stable manifold of a saddle on $\hat{\mathcal{M}}_\varepsilon$. For an illustration of proposition 7.2, see figure 10.

PROOF OF PROPOSITION 7.2: As in the proof of the previous proposition, we again show only the first part of this proposition. Let us choose a small enough δ , and consider the region $\delta/2 \leq \|x - X(I)\| \leq \delta$. The previous proposition implies that the local unstable manifold $W_{loc}^u(O_{1,0}(\lambda))$ and the limiting intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ intersect transversely inside the piece of the local unstable manifold $W_{loc}^u(\hat{\mathcal{M}})$ that satisfies the inequalities $\delta/2 \leq \|x - X(I)\| \leq \delta$.

Now, for small positive ε , the piece of the homoclinic intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ inside $\delta/2 \leq \|x - X(I)\| \leq \delta$ is $\mathcal{O}(\sqrt{\varepsilon})$ away from the corresponding piece of the surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$, and is contained in the local unstable manifold $W_{loc}^u(\hat{\mathcal{M}}_\varepsilon)$. By persistence of transverse intersections inside the manifold $W_{loc}^u(\hat{\mathcal{M}}_\varepsilon)$, the pieces of the surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ and the local unstable manifold $W_{loc}^u(O_{1,\varepsilon}(\lambda))$ that lie inside the region $\delta/2 \leq \|x - X(I)\| \leq \delta$ still intersect each other. By (local) invariance, this intersection must take place along a segment of

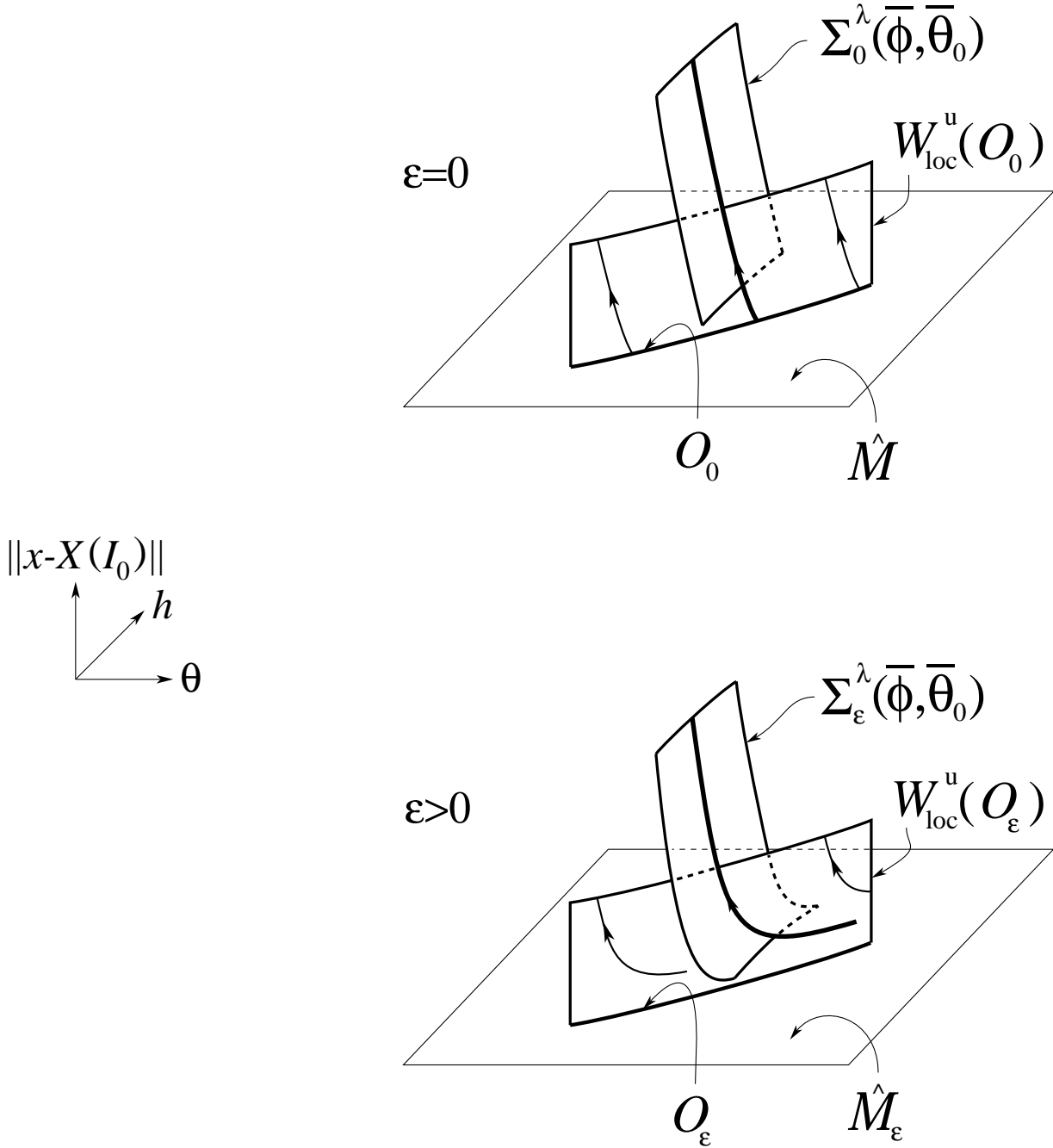


Figure 10: The local unstable manifold $W_{loc}^u(O_{1,0}(\lambda))$ of the curve $O_{1,0}(\lambda)$ and the limiting homoclinic intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ intersect transversely inside the local unstable manifold $W_{loc}^u(\hat{\mathcal{M}})$ of the annulus $\hat{\mathcal{M}}$. Therefore, the local unstable manifold $W_{loc}^u(O_{1,\varepsilon}(\lambda))$ of the orbit segment $O_{1,\varepsilon}(\lambda)$ and the homoclinic intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ must intersect transversely inside the local unstable manifold $W_{loc}^u(\hat{\mathcal{M}}_\varepsilon)$ of the annulus $\hat{\mathcal{M}}_\varepsilon$ for small positive ε .

an orbit. This orbit segment is $\mathcal{O}(\sqrt{\varepsilon})$ close to the segment of the intersection orbit of the unperturbed local unstable manifold $W_{loc}^u(O_{1,0}(\lambda))$ and the limiting intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ that lies in the region $\delta/2 \leq \|x - X(I)\| \leq \delta$.

The whole orbit, $a_{1,\varepsilon}^\lambda(t)$, which we obtain from this perturbed orbit segment by evolving it in forward and backward time, must be contained in the unstable manifold $W^u(O_{1,\varepsilon}(\lambda))$ by the invariance of this manifold. The $\mathcal{O}(\sqrt{\varepsilon})$ proximity for finite times of the trajectories on this orbit and the intersection orbit of the unperturbed unstable manifold $W^u(O_{1,0}(\lambda))$ and the limiting intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ now follows by proposition 4.2. \square

The preceding proposition now renders the following:

PROOF OF THEOREM 1: By the previous proposition, there exist two particular orbits on the homoclinic intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$, one forward asymptotic to either the periodic orbit $O_{2,\varepsilon}(\lambda)$ or the saddle $s_{2,\varepsilon}(\lambda)$, and another backward asymptotic to either the periodic orbit $O_{1,\varepsilon}(\lambda)$ or the saddle $s_{1,\varepsilon}(\lambda)$. Also, by the previous proposition and because of equation (3.8), for $\lambda > \bar{\lambda}$, the h coordinate of any point on one of these orbits is always larger than the h coordinate of the corresponding point on the other orbit at the same value of θ . For $\lambda < \bar{\lambda}$, the roles are reversed. Therefore, at some $\lambda = \lambda(\varepsilon)$ near $\lambda = \bar{\lambda}$ with $\lambda(0) = \bar{\lambda}$, the two orbits must pass through each other, and thus form a heteroclinic orbit connecting either the periodic orbit $O_{1,\varepsilon}(\lambda(\varepsilon))$ or the saddle $s_{1,\varepsilon}(\lambda(\varepsilon))$ to either the periodic orbit $O_{2,\varepsilon}(\lambda(\varepsilon))$ or the saddle $s_{2,\varepsilon}(\lambda(\varepsilon))$, as claimed. \square

An analogous theorem can be proven in the Hamiltonian case. There, two of the three main ingredients of the proof are again propositions 7.1 and 7.2. However, the transversality condition (3.8) must be dropped, and condition (3.7) is now identical to setting the first component of the Melnikov vector equal to zero. In the proof, an energy argument must be used to show the existence of a heteroclinic connection instead of the transversality argument following from the condition (3.8) used here. Moreover, in the Hamiltonian case, the homoclinic connection will exist for all λ close enough to $\lambda = \bar{\lambda}$. For details, see ref. [40].

In order to prove theorem 2, we need to show the existence of the heteroclinic orbit in question, as well as the fact that the manifolds $W^u(O_{1,\varepsilon}(\lambda))$ and $W^s(\hat{\mathcal{M}}_\varepsilon)$ intersect transversely along this heteroclinic orbit. The argument proceeds as follows:

PROOF OF THEOREM 2: Let λ be close to $\lambda = \bar{\lambda}$. Then proposition 7.2 ensures that there exists an orbit that is contained in both the unstable manifold $W^u(O_{1,\varepsilon}(\lambda))$ and the intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$. Let $a_{1,\varepsilon}^\lambda(t)$ be a trajectory on this orbit, and let $a_{1,0}^\lambda(t)$ be a

trajectory on its unperturbed counterpart that starts $\mathcal{O}(\sqrt{\varepsilon})$ away from $a_{1,\varepsilon}^\lambda(t)$ at $t = 0$. Let $T > 0$ be large enough so that both $a_{1,0}^\lambda(t)$ and $a_{1,\varepsilon}^\lambda(t)$ return inside the neighborhood U_δ at the time $t = T$. By proposition 4.2, the points $a_{1,0}^\lambda(T)$ and $a_{1,\varepsilon}^\lambda(T)$ are at most $\mathcal{O}(\sqrt{\varepsilon})$ apart. The stable fiber passing through the point $a_{1,\varepsilon}^\lambda(T)$ is $\mathcal{O}(\sqrt{\varepsilon})$ close to the stable fiber passing through the point $a_{1,0}^\lambda(T)$ by proposition 6.1. Therefore, the base points of these two fibers are $\mathcal{O}(\sqrt{\varepsilon})$ close, as well. But the base point of the fiber that passes through the point $a_{1,0}^\lambda(T)$ is precisely the point $(h, \theta) = (0, \bar{\theta}_0(I_0, \lambda) + \Delta\theta_+(\bar{\phi}(I_0, \lambda)))$. Thus, for all small enough ε , the base point of the fiber through $a_{1,\varepsilon}^\lambda(T)$ must be contained in the basin of attraction \mathcal{B}_ε , which proves the existence part of the theorem.

To show the transversality of the intersection of the unstable manifold $W^u(O_{1,\varepsilon}(\lambda))$ with the stable manifold $W^s(\hat{\mathcal{M}}_\varepsilon)$ of the perturbed annulus $\hat{\mathcal{M}}_\varepsilon$ along the orbit $a_\varepsilon(t)$, we first recall that proposition 7.1 implies that the intersection of the homoclinic surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ and the unstable manifold $W^u(O_{1,\varepsilon}(\lambda))$ is transverse inside the unstable manifold $W^u(\hat{\mathcal{M}}_\varepsilon)$ of $\hat{\mathcal{M}}_\varepsilon$. We also recall that, by proposition 4.3, the manifolds $W^s(\hat{\mathcal{M}}_\varepsilon)$ and $W^u(\hat{\mathcal{M}}_\varepsilon)$ intersect along the surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ transversely in the full phase space. This clearly implies that, since $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ is contained in the stable manifold $W^s(\hat{\mathcal{M}}_\varepsilon)$, this stable manifold and the unstable manifold $W^u(O_{1,\varepsilon}(\lambda))$ must intersect transversely along the heteroclinic orbit $a_\varepsilon(t)$ in the full phase space. Now, either the stable manifold $W^s(c_\varepsilon(\lambda))$ of the equilibrium $c_\varepsilon(\lambda)$ or the stable manifold $W^s(O_{2,\varepsilon}(\lambda))$ of the periodic orbit $O_{2,\varepsilon}(\lambda)$ are neighborhoods of the orbit $a_{1,\varepsilon}^\lambda(T)$ inside the stable manifold $W^s(\hat{\mathcal{M}}_\varepsilon)$ of the annulus $\hat{\mathcal{M}}_\varepsilon$. Therefore, the above transversality argument holds for the manifolds $W^s(c_\varepsilon(\lambda))$ or $W^s(O_{2,\varepsilon}(\lambda))$ in place of the manifold $W^s(\hat{\mathcal{M}}_\varepsilon)$, which concludes our proof. \square

Finally, we prove Theorem 3. A different proof of a special case of this theorem with $x \in \mathbb{R}^2$ appeared in [30]. The present proof is included in this paper in order to show how the result of [30] fits in the more general framework that leads at once to all three theorems, 1, 2, and 3, and also to extend the proof of [30] to the case when $x \in \mathbb{R}^{2n}$ with $n > 1$. , 1, 2, and 3.

PROOF OF THEOREM 3: First recall that, in the inner limit, the unstable manifold $W^u(\hat{\mathcal{M}})$ of the annulus $\hat{\mathcal{M}}$ is parametrized by t, h, ϕ and θ_0 in the expression

$$(x^h(t, I_0, \phi), h, \theta^h(t, I_0, \phi) + \theta_0). \quad (7.1)$$

The unstable manifold $W^u(c_0(\lambda))$ of the point $c_0(\lambda)$ is parametrized by t and ϕ in the expression obtained by choosing $h = 0$ and $\theta_0 = \theta(c_0(\lambda)) + \Delta\theta_-(\phi)$ in formula (7.1). Likewise, the limiting homoclinic intersection surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ is parametrized by t and h in the

expression obtained by choosing $\phi = \bar{\phi}(I_0, \lambda)$ and $\theta_0 = \bar{\theta}_0(I_0, \lambda)$ in formula (7.1).

At $\lambda = \bar{\lambda}$, the manifold $W^u(c_0(\lambda))$ and the surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ intersect along a unique orbit given by the expression

$$\left(x^h(t, I_0, \bar{\phi}(I_0, \bar{\lambda})), 0, \theta^h(t, I_0, \bar{\phi}(I_0, \bar{\lambda})) + \bar{\theta}_0(I_0, \bar{\lambda})\right), \quad (7.2)$$

because, by equation (3.9), we must have $\bar{\theta}_0(I_0, \bar{\lambda}) - \Delta\theta_-(\bar{\phi}(I_0, \bar{\lambda})) = \theta(c_0(\bar{\lambda}))$. Moreover, by formula (3.10), the passage of the manifold $W^u(c_0(\lambda))$ and the surface $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ through each other along the orbit (7.2) as λ passes through $\lambda = \bar{\lambda}$ is transverse inside the unstable manifold $W^u(\hat{\mathcal{M}})$.

Recall now the definition of the neighborhood U_δ of the annuli $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}_\varepsilon$, whose points satisfy the formula $\|x - X(I)\| < \delta$. Outside of a smaller neighborhood, say $U_{\delta/2}$, of the annuli $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}_\varepsilon$, the homoclinic intersection surfaces $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ and $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ are $\mathcal{O}(\sqrt{\varepsilon})$ close to each other. Moreover, the local unstable manifold $W_{loc}^u(c_\varepsilon(\lambda))$ varies smoothly with λ and $\sqrt{\varepsilon}$ down to and including $\varepsilon = 0$ inside the neighborhood U_δ by Proposition 6.3. Hence, it follows from the discussion in the previous paragraph that, in the region $\delta/2 < \|x - X(I)\| < \delta$, the manifold $W_{loc}^u(c_\varepsilon(\lambda))$ and the homoclinic intersection surface $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ must pass through each other transversely inside the local unstable manifold $W_{loc}^u(\hat{\mathcal{M}}_\varepsilon)$ as λ varies through some $\lambda = \lambda(\varepsilon)$. The function $\lambda(\varepsilon)$ varies smoothly with $\sqrt{\varepsilon}$, and its value at $\varepsilon = 0$ is $\bar{\lambda}$. Moreover, the intersection of $W_{loc}^u(c_\varepsilon(\lambda))$ and $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ at $\lambda = \lambda(\varepsilon)$ inside the region $\delta/2 < \|x - X(I)\| < \delta$ takes place along an orbit segment that is $\mathcal{O}(\sqrt{\varepsilon})$ close to an appropriate segment of the orbit (7.2).

Let $a_0(t)$ and $a_\varepsilon(t)$ be two trajectories that start $\mathcal{O}(\sqrt{\varepsilon})$ away from each other at $t = 0$, and lie on the intersections of the unperturbed and the perturbed unstable manifolds, $W^u(c_0(\bar{\lambda}))$ and $W^u(c_\varepsilon(\lambda(\varepsilon)))$ with the homoclinic surfaces $\Sigma_0^\lambda(\bar{\phi}, \bar{\theta}_0)$ and $\Sigma_\varepsilon^\lambda(\bar{\phi}, \bar{\theta}_0)$ at $\lambda = \bar{\lambda}$ and $\lambda = \lambda(\varepsilon)$, respectively. Since $\lambda(\varepsilon)$ and $\bar{\lambda}$ are at most $\mathcal{O}(\sqrt{\varepsilon})$ apart for small ε , we can proceed as in the proof of theorem 2 to show that these trajectories are $\mathcal{O}(\sqrt{\varepsilon})$ close to each other for all times up to including some large enough $T > 0$, and that the stable fibers passing through the respective points $a_\varepsilon(T)$ and $a_0(T)$ are also at most $\mathcal{O}(\sqrt{\varepsilon})$ apart. Thus, as in the proof of theorem 2, we conclude that the trajectory $a_\varepsilon(t)$ is attracted to the same object as the points in the set \mathcal{B}_ε , which proves the theorem. \square

We now make a remark about the uniqueness of the heteroclinic orbits discussed in theorems 1, 2, and 3. This remark is in place because the families of stable and unstable fibers described in proposition 6.1 that foliate the manifolds $W^s(\hat{\mathcal{M}}_\varepsilon)$ and $W^u(\hat{\mathcal{M}}_\varepsilon)$ need not be unique; see ref. [37]. However, for $\varepsilon > 0$, the equilibria and periodic orbits that

the heteroclinic orbits in question connect are by assumption hyperbolic, and thus unique. Their stable and unstable manifolds are therefore also unique, and so must be the heteroclinic orbits that arise as the intersections of these manifolds.

8 An Example

We consider a four-parameter family of problems in which a Duffing oscillator is coupled to an anharmonic oscillator, described by the system of equations

$$\dot{p} = \mu^2 q(I - q^2) - \varepsilon \alpha p, \quad (8.1a)$$

$$\dot{q} = p, \quad (8.1b)$$

$$\dot{I} = -\varepsilon I \sin \theta - \varepsilon \beta I - \varepsilon \gamma p^2, \quad (8.1c)$$

$$\dot{\theta} = I - 1 - \frac{1}{2} \mu^2 q^2 - \varepsilon \cos \theta, \quad (8.1d)$$

where μ , α , β , and γ are positive parameters, and $\varepsilon \ll 1$ a small positive parameter. This system is of the form

$$\begin{aligned} \dot{p} &= -\frac{\partial H(p, q, I, \theta)}{\partial q} - \varepsilon \alpha p, & \dot{q} &= \frac{\partial H(p, q, I, \theta)}{\partial p}, \\ \dot{I} &= -\frac{\partial H(p, q, I, \theta)}{\partial \theta} - \varepsilon \beta I - \varepsilon \gamma p^2, & \dot{\theta} &= \frac{\partial H(p, q, I, \theta)}{\partial I}, \end{aligned}$$

where

$$\begin{aligned} H(p, q, I, \theta) &= H_0(p, q, I) + \varepsilon H_1(p, q, I, \theta) \\ &= \frac{1}{2} I^2 - I + \frac{1}{2} p^2 - \frac{1}{2} \mu^2 q^2 (I - \frac{1}{2} q^2) - \varepsilon I \cos \theta. \end{aligned} \quad (8.2)$$

It is easily seen that equations (8.1) fall into the same category as (2.1).

The unperturbed equations corresponding to (8.1) are

$$\dot{p} = \mu^2 q(I - q^2), \quad (8.3a)$$

$$\dot{q} = p, \quad (8.3b)$$

$$\dot{I} = 0, \quad (8.3c)$$

$$\dot{\theta} = I - 1 - \frac{1}{2} \mu^2 q^2, \quad (8.3d)$$

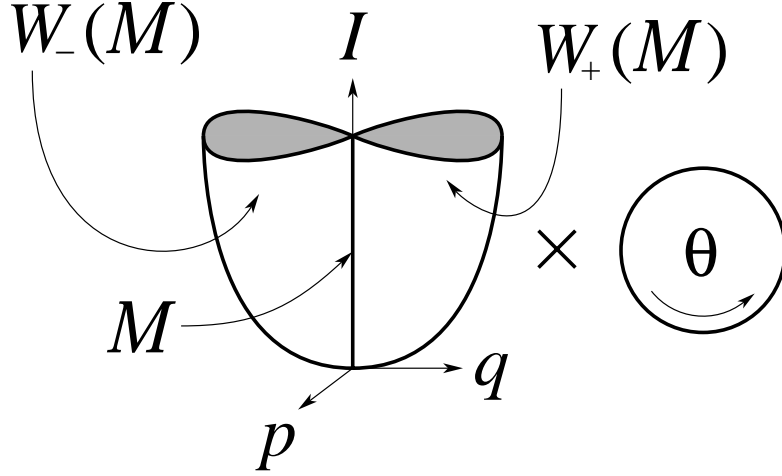


Figure 11: The unperturbed invariant annulus, \mathcal{M} , and its homoclinic manifolds, $W_+(\mathcal{M})$ and $W_-(\mathcal{M})$ for the Duffing oscillator-harmonic oscillator example.

that can be derived from the unperturbed Hamiltonian

$$H_0(p, q, I) = \frac{1}{2}I^2 - I + \frac{1}{2}p^2 - \frac{1}{2}\mu^2 q^2(I - \frac{1}{2}q^2). \quad (8.4)$$

The unstable invariant annulus \mathcal{M} is located at $(p, q) = (0, 0)$, and can be bounded by any I_1 and I_2 , with $0 < I_1 < 1 < I_2$. It is foliated by periodic orbits $p = q = 0$, $I = \text{constant}$, $\theta = (I - 1)t + \theta_0$. The orbit at $I = 1$ is a circle of equilibria, and clearly, the frequency $I - 1$ passes through zero transversely there, so that the resonance assumption 3 is satisfied. The annulus \mathcal{M} is connected to itself by a pair of three-dimensional homoclinic manifolds, $W_+(\mathcal{M})$, and $W_-(\mathcal{M})$, as is shown in figure 11. The manifolds $W_+(\mathcal{M})$, and $W_-(\mathcal{M})$ are parametrized by t , I and θ_0 in the homoclinic solutions

$$p = p^h(t, I) = \mp\sqrt{2}\mu I \operatorname{sech}(\mu\sqrt{I}t) \tanh(\mu\sqrt{I}t), \quad (8.5a)$$

$$q = q^h(t, I) = \pm\sqrt{2I} \operatorname{sech}(\mu\sqrt{I}t), \quad (8.5b)$$

$$I = I, \quad (8.5c)$$

$$\theta = \theta^h(t, I) + \theta_0 = (I - 1)t - \mu\sqrt{I} \tanh(\mu\sqrt{I}t) + \theta_0. \quad (8.5d)$$

(In this example, the number of dimensions is too small to require the additional parameters ϕ .) From (8.5d) we find the angle difference $\Delta\theta$ between the end points of any heteroclinic orbit connecting pairs of equilibria on \mathcal{M} at $I = 1$ to be $\Delta\theta = -2\mu$.

Even for nonzero ε , the set $p = q = 0$ is invariant. Thus, we can take the perturbed annulus \mathcal{M}_ε to be the same as the annulus \mathcal{M} . At the resonance, $I = 1$, the Melnikov

function, $M(I, \theta_0, \alpha, \beta, \gamma)$, can be computed explicitly [49]. This is because the integrand in formula (2.6) in this case reduces to

$$-\frac{dH_1}{dt}(p, q, 1, \theta) - \alpha p^2 + \frac{1}{2}\mu^2\beta q^2 + \frac{1}{2}\mu^2\gamma p^2 q^2,$$

where $p = p^h(t, 1)$, $q = q^h(t, 1)$, and $\theta = \theta^h(t, 1) + \theta_0$. Thus, the Melnikov function becomes

$$\begin{aligned} M(1, \theta_0, \alpha, \beta, \gamma) &= H_1(0, 0, 1, \theta_0 + \mu) - H_1(0, 0, 1, \theta_0 - \mu) - \frac{4}{3}\alpha\mu + 2\beta\mu + \frac{8}{15}\gamma\mu^3 \\ &= -\cos(\theta_0 + \mu) + \cos(\theta_0 - \mu) - \frac{4}{3}\alpha\mu + 2\beta\mu + \frac{8}{15}\gamma\mu^3 \\ &= 2\sin\mu\sin\theta_0 - \frac{4}{3}\alpha\mu + 2\beta\mu + \frac{8}{15}\gamma\mu^3, \end{aligned} \tag{8.6}$$

and is the same on both homoclinic manifolds, $W_+(\mathcal{M})$ and $W_-(\mathcal{M})$.

When μ is not a multiple of π , this Melnikov function has *transverse* zeros in θ_0 at $\theta_0 = \bar{\theta}_{0,1}$ and $\theta_0 = \bar{\theta}_{0,2} = \pi - \bar{\theta}_{0,1}$, provided that

$$\left| \frac{\mu}{\sin\mu} \left(\frac{2}{3}\alpha - \beta - \frac{4}{15}\gamma\mu^2 \right) \right| < 1. \tag{8.7}$$

For all admissible α , β , and γ , the stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ intersect transversely along two symmetric pairs of two-dimensional homoclinic surfaces, $\Sigma_{\pm, \varepsilon}^{\alpha, \beta, \gamma}(\bar{\theta}_{0,1})$ and $\Sigma_{\pm, \varepsilon}^{\alpha, \beta, \gamma}(\bar{\theta}_{0,2})$.

The restricted system, (3.2), at the resonance at $I = 1$ for this example is

$$h' = -(1 + \sqrt{\varepsilon}h)\sin\theta - \beta(1 + \sqrt{\varepsilon}h), \quad \theta' = h - \sqrt{\varepsilon}\cos\theta, \tag{8.8}$$

and the limiting outer system is

$$h' = -\sin\theta - \beta, \quad \theta' = h. \tag{8.9}$$

The rescaled Hamiltonian is

$$\mathcal{H}(h, \theta) = \frac{1}{2}h^2 - \cos\theta + \beta\theta, \tag{8.10}$$

which is the Hamiltonian of the pendulum subjected to a constant torque.

The phase portrait of the rescaled $h - \theta$ phase cylinder \mathcal{M}_0 of the equations (8.9) is shown in figure 12. There are two equilibria on this phase cylinder, a center, c_0 , at $(h, \theta) = (0, -\arcsin\beta)$, and a saddle, s_0 , at $(h, \theta) = (0, -\pi + \arcsin\beta)$. The two branches of the

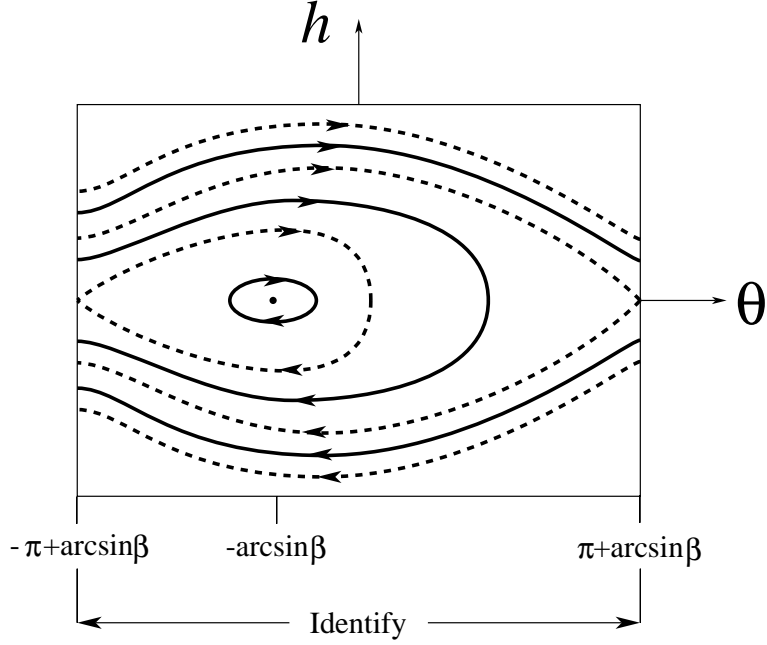


Figure 12: The phase portrait of the $h - \theta$ cylinder $\hat{\mathcal{M}}$ at the resonance. Broken lines represent the stable and unstable manifolds, $\mathcal{W}^s(s_0)$ and $\mathcal{W}^u(s_0)$, of the saddle s_0 .

stable and unstable manifolds, $\mathcal{W}^s(s_0)$ and $\mathcal{W}^u(s_0)$, to the right of the saddle s_0 coincide to form a separatrix that encloses a family of periodic orbits nested around the center. The two branches of the manifolds $\mathcal{W}^s(s_0)$ and $\mathcal{W}^u(s_0)$ to the left of the saddle s_0 wind around the cylinder \mathcal{M}_0 towards $h = +\infty$ and $h = -\infty$, respectively. For small positive $\sqrt{\varepsilon}$, the saddle s_0 persists as a saddle, s_ε , the center c_0 becomes a sink, c_ε , and the separatrix breaks. The top branch of the unstable manifold, $\mathcal{W}^u(s_\varepsilon)$, of the perturbed saddle s_ε falls into the sink c_ε . No periodic orbits are left in this system, and all the points that lie in any compact domain that is all contained inside the unperturbed separatrix asymptote to the sink c_ε .

The inner system is

$$\dot{p} = \mu^2 q(1 - q^2), \quad (8.11a)$$

$$\dot{q} = p, \quad (8.11b)$$

$$\dot{h} = 0, \quad (8.11c)$$

$$\dot{\theta} = -\frac{1}{2}\mu^2 q^2. \quad (8.11d)$$

In the phase space of this system, the two symmetric pairs of homoclinic intersection surfaces, $\Sigma_{\pm, \varepsilon}^{\alpha, \beta, \gamma}(\bar{\theta}_{0,1})$ and $\Sigma_{\pm, \varepsilon}^{\alpha, \beta, \gamma}(\bar{\theta}_{0,2})$ (when the inequality (8.7) shows that they exist), collapse smoothly onto the pairs of surfaces, $\Sigma_{\pm, 0}^{\alpha, \beta, \gamma}(\bar{\theta}_{0,1})$ and $\Sigma_{\pm, 0}^{\alpha, \beta, \gamma}(\bar{\theta}_{0,2})$, parametrized by the expressions (8.5) with $I = 1$, $\theta_0 = \bar{\theta}_{0,1}$ or $\bar{\theta}_{0,2}$, and arbitrary h .

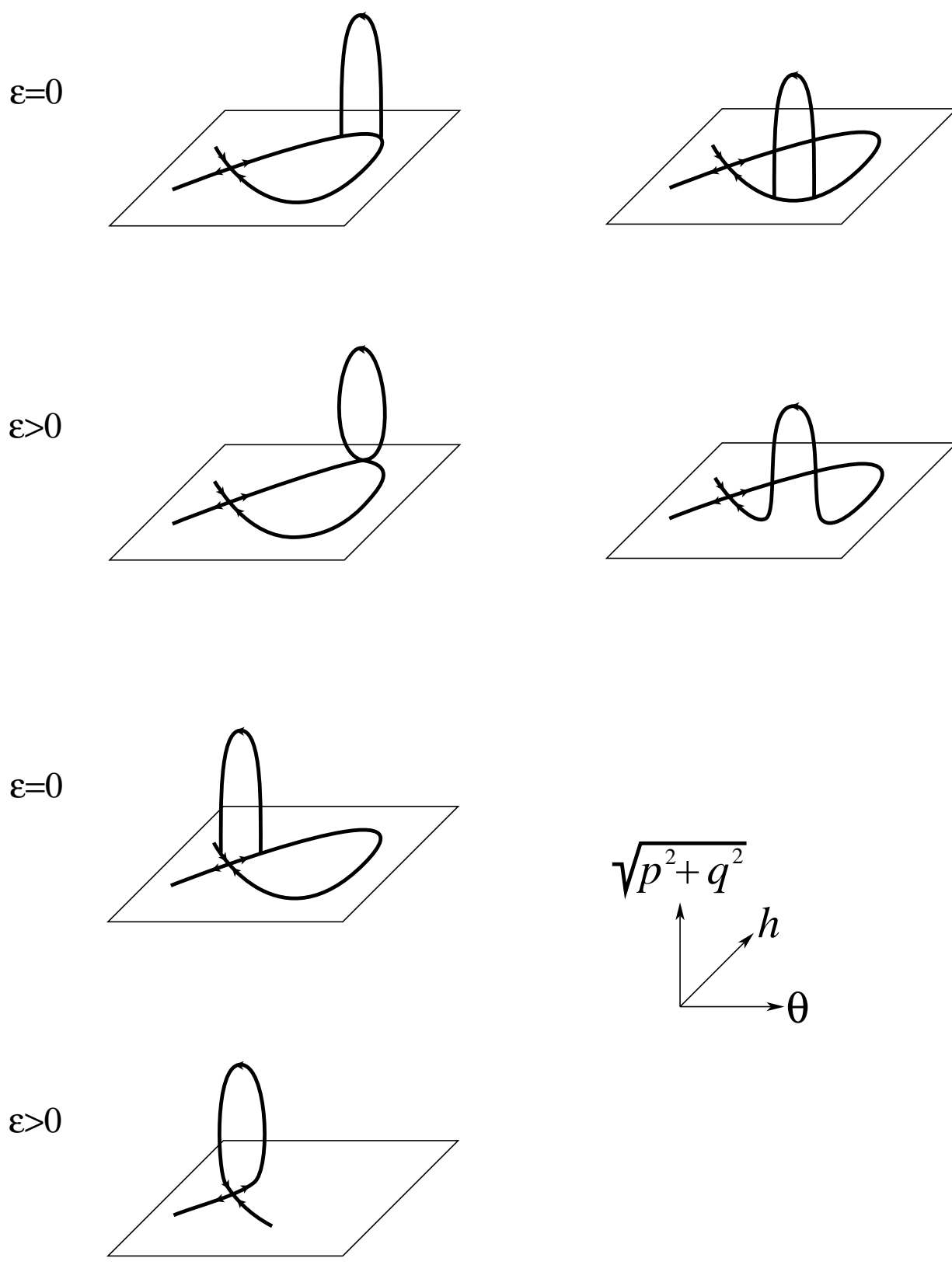


Figure 13: The three types of orbits homoclinic to the saddle s_ε , whose existence follows from theorem 1.

We now demonstrate that this example satisfies the conditions of theorems 1, 2, and 3. In order to do so, we consider the case when $\mu \ll \beta < 1$. We assume that $\tilde{\gamma} = \mu^2 \gamma = \mathcal{O}(1)$, and let α play the role of the parameter λ . We will show that, for appropriately chosen α and $\tilde{\gamma}$, orbits homoclinic to the saddle s_ε , orbits connecting s_ε to the sink c_ε , and orbits homoclinic to the sink c_ε exist. In fact, due to the symmetry of the problem, all such orbits always occur in pairs: one on the surface $\Sigma_{+,\varepsilon}^{\alpha,\beta,\gamma}(\bar{\theta}_{0,1})$ and the other one on $\Sigma_{-,\varepsilon}^{\alpha,\beta,\gamma}(\bar{\theta}_{0,2})$.

First, we use theorem 1 to find pairs of orbits homoclinic to the saddle s_ε . In fact, three different types of such homoclinic orbits exist. They are shown in figure 13, and in order to prove their existence, we proceed as follows: From (8.6), we recall that the equation for any zero, $\bar{\theta}_0$, of the Melnikov function satisfies the equation

$$2 \sin \mu \sin \bar{\theta}_0 - \frac{4}{3} \alpha \mu + 2 \beta \mu + \frac{8}{15} \tilde{\gamma} \mu = 0.$$

Moreover, formula (3.7) for this example reads

$$\begin{aligned} \frac{1}{2} h^2(\bar{\theta}_0 + \mu) - \frac{1}{2} h^2(\bar{\theta}_0 - \mu) &= \cos(\bar{\theta}_0 + \mu) - \beta \mu - \cos(\bar{\theta}_0 - \mu) - \beta \mu \\ &= -2 \sin \mu \sin \bar{\theta}_0 - 2 \beta \mu \\ &= -\frac{4}{3} \alpha \mu + \frac{8}{15} \tilde{\gamma} \mu \\ &= 0. \end{aligned}$$

Thus, in order to find an orbit homoclinic to the point s_ε , we must solve the equations

$$\sin \bar{\theta}_0 = -\beta \frac{\mu}{\sin \mu} = -\beta + \mathcal{O}(\mu^2),$$

and

$$5\alpha - 2\tilde{\gamma} = 0.$$

We thus obtain

$$\bar{\theta}_{0,1} = -\arcsin \beta + \mathcal{O}(\mu^2)$$

and

$$\bar{\theta}_{0,2} = -\pi + \arcsin \beta + \mathcal{O}(\mu^2).$$

This implies that the line $\theta = \bar{\theta}_{0,1}$ passes an $\mathcal{O}(\mu^2)$ distance away from the center c_0 , and that the line $\theta = \bar{\theta}_{0,2}$ passes an $\mathcal{O}(\mu^2)$ distance away from the saddle s_0 , so that the desired intersections exist, and are at the same height, h , whenever $\alpha = 2\tilde{\gamma}/5$.

It can be shown that, in general, if $\mu = n\pi + \delta$, with some nonnegative integer n and $|\delta| \ll \beta < 1$, we obtain orbits homoclinic to the saddle s_ε that wind n times around the cylinder \mathcal{M}_ε before returning to it. (See ref. [35].)

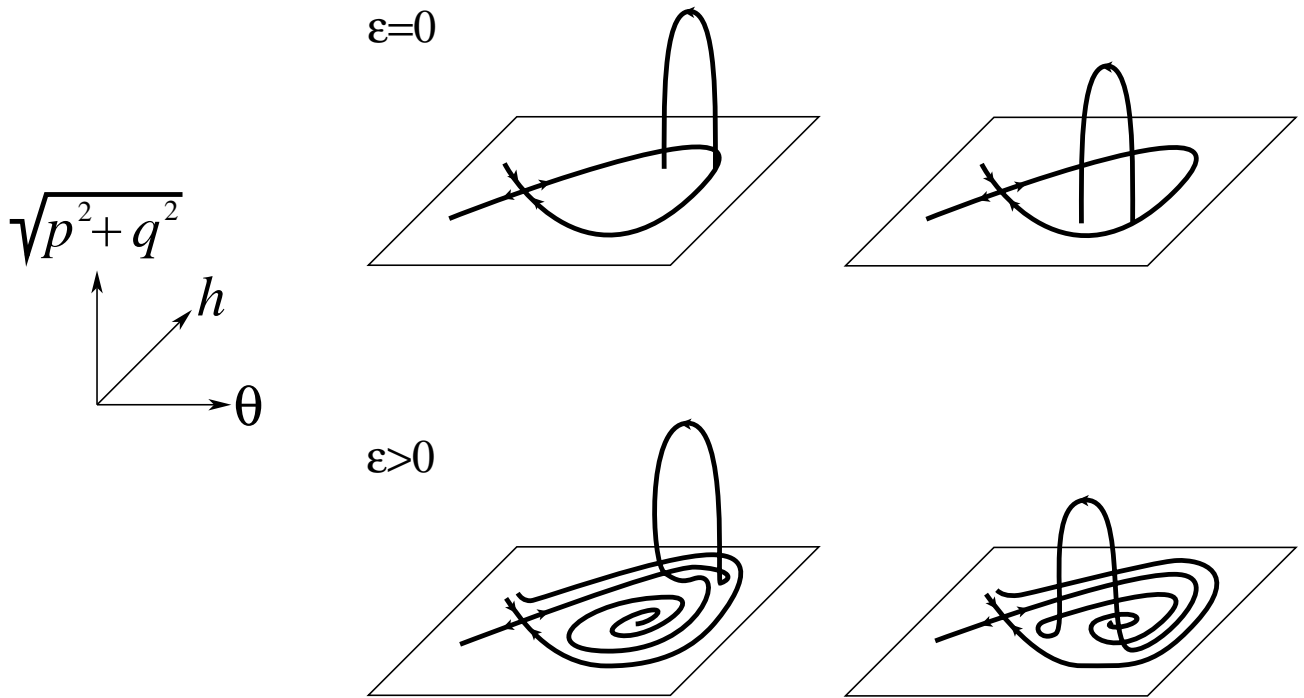


Figure 14: The two types of heteroclinic orbits connecting the saddle s_ε and the spiral-saddle c_ε , whose existence follows from theorem 3.

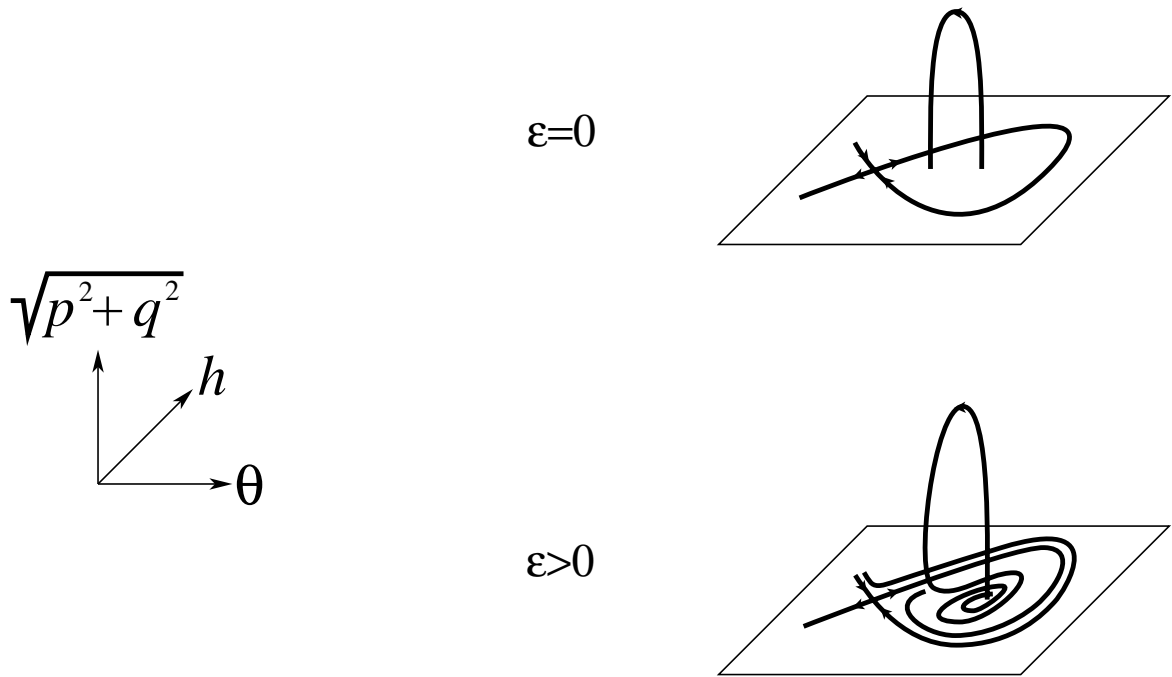


Figure 15: An orbit homoclinic to the spiral-saddle c_ε , whose existence follows from theorem 2.

If we choose α and $\tilde{\gamma}$ so that $5\alpha - 2\tilde{\gamma} > 0$ but sufficiently small, then the hypotheses of theorem 2 are satisfied, and there exist two pairs of connections between the saddle s_ε and the sink c_ε , as shown in figure 14.

Finally, we show the existence of orbits homoclinic to the point c_ε , shown in figure 15. One condition for a pair of such orbits to exist is that the line $\theta = \bar{\theta}_0 + \mu$ must pass through the unperturbed equilibrium c_0 . This happens when $\cos(\bar{\theta}_0 + \mu) = \sqrt{1 - \beta^2}$ and $\sin(\bar{\theta}_0 + \mu) = -\beta$, hence formula (8.6) implies the equation

$$-\sqrt{1 - \beta^2} + \sqrt{1 - \beta^2} \cos 2\mu - \beta \sin 2\mu - \frac{4}{3}\alpha\mu + 2\beta\mu + \frac{8}{15}\tilde{\gamma}\mu = 0,$$

which, when $\mu \ll \beta < 1$, yields

$$\alpha = \frac{2}{5}\tilde{\gamma} - \frac{3}{2}\mu\sqrt{1 - \beta^2} + \mathcal{O}(\mu^2).$$

The second condition is that the point $(h, \theta) = (0, \bar{\theta}_0 - \mu)$ be contained inside the separatrix that encircles the equilibrium point at $(h, \theta) = (0, \bar{\theta}_0 + \mu)$, which is clearly satisfied for $\mu \ll \beta < 1$. When both of these conditions are satisfied, it follows from theorem 3 that a pair of orbits homoclinic to the equilibrium c_ε exists. These orbits are of the so-called Šilnikov type [50]; the chaotic dynamics created by such a pair are discussed, for instance in ref. [30].

In conclusion, even this simple example shows the richness of the various homoclinic orbits that may emerge under perturbation from orbits homoclinic to an unstable circle of equilibria that breaks up into a resonance band. Furthermore, this example also shows the ease with which theorems 1, 2, and 3 can be applied to specific situations, and thus reveals the potential power of the method for finding orbits homoclinic to resonance bands in solving physical and engineering problems.

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