Homoclinic Orbits in Adiabatic Systems: a Short Review

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Abstract

This paper reviews a simple geometric theory that describes the existence criteria for and the consequences of transverse homoclinic orbits in a large class of periodically and adiabatically driven planar Hamiltonian systems. In particular, the geometric meaning of the Melnikov function and a derivation of an approximate expression for the flux of phase points through the associated homoclinic tangle are explained.

1 Introduction

In this paper, we review a geometric theory of Smale-horseshoe-type chaotic dynamics due to transverse homoclinic orbits in a large class of periodically and adiabatically driven planar Hamiltonian systems as presented in [1]. These are systems that can be derived from planar Hamiltonians of the form $H(p, q, \varepsilon t)$ that are 2π periodic in εt, where ε is a small parameter $0 < \varepsilon \ll 1$. Solutions to these systems are the solutions of the equations

$$\dot{p} = -\frac{\partial H}{\partial q}(p, q, z), \quad \dot{q} = \frac{\partial H}{\partial p}(p, q, z), \quad \dot{z} = \varepsilon \tag{1}$$

with initial conditions $z = 0$ at $t = 0$.

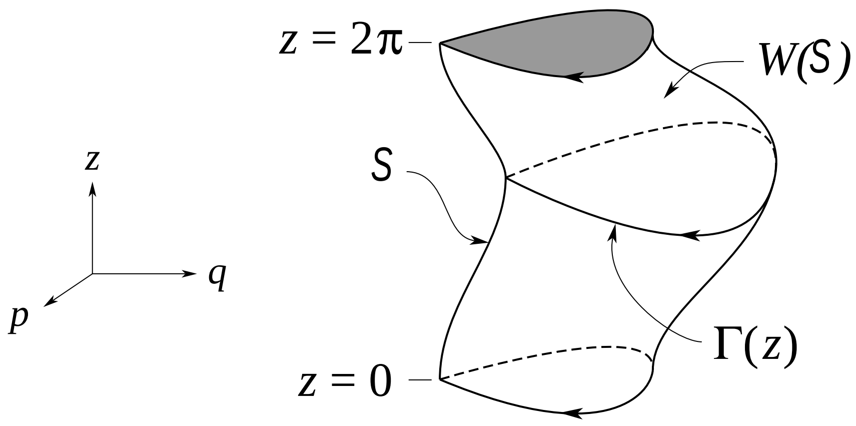

Figure 1: The homoclinic manifold $W(\mathcal{S})$.

The class under review can be characterized in the following way: For $\varepsilon = 0$, equations (1) become a one-parameter family of planar Hamiltonian systems

$$\dot{p} = -\frac{\partial H}{\partial q}(p, q, z), \quad \dot{q} = \frac{\partial H}{\partial p}(p, q, z), \quad \dot{z} = 0, \tag{2}$$

which we call the unperturbed system, as opposed to the perturbed system (1). We assume that

Assumption 1 *For every z, with $0 \leq z \leq 2\pi$, the planar system for p and q possesses a hyperbolic equilibrium at $(p, q) = (P(z), Q(z))$, connected to itself by a homoclinic orbit $\Gamma(z)$.*

This assumption implies, in particular, that since the equilibria at $(P(z), Q(z))$ are hyperbolic for all values of z, they form a smooth string $\mathcal{S}$ of equilibria in the $p-q-z$ space. This string is periodic in z with period 2π, and is connected to itself by the smooth two-dimensional homoclinic manifold $W(\mathcal{S})$, which is just the union of all the homoclinic orbits $(\Gamma(z), z)$, see Figure 1.

The purpose of this note is to review a simple geometric theory that describes which of the unperturbed homoclinic orbits $\Gamma(z)$ of the system (2) survive for small values of ε, and what the consequences of their survival are for the existence of Smale-horseshoe-type chaotic dynamics and transport of phase points in the extended phase space. This review is given in Theorems 1 and 2 below. Theorem 1 addresses a geometric criterion for the survival of unperturbed homoclinic orbits, and Theorem 2 discusses the consequences of this criterion for the transport of points through the extended phase space.

The rest of this note is organized as follows. In Section 2, we discuss Theorem 1, the background that leads to it, and its corollaries. In Section 3, we discuss Theorem 2 and the background that leads to it. In Sections 4 and 5 we prove the two theorems, respectively.

2 The Melnikov Function and its Geometric Interpretation

In this section, we develop a simple geometric criterion to determine which of the unperturbed homoclinic orbits $\Gamma(z)$ survives for small nonzero ε.

To begin with, for small nonzero ε, the string of equilibria $\mathcal{S}$ persists as a hyperbolic periodic orbit $\mathcal{S}_\varepsilon$, a distance $\mathcal{O}(\varepsilon)$ away. The homoclinic manifold $W(\mathcal{S})$ splits into two manifolds, the stable manifold $W^s(\mathcal{S}_\varepsilon)$ and the unstable manifold $W^u(\mathcal{S}_\varepsilon)$ of the periodic orbit $\mathcal{S}_\varepsilon$. Transverse intersections of the manifolds $W^s(\mathcal{S}_\varepsilon)$ and $W^u(\mathcal{S}_\varepsilon)$ occur along orbits homoclinic to the orbit $\mathcal{S}_\varepsilon$ that have survived for nonzero ε, and imply existence of Smale-horseshoe-type chaotic dynamics and phase-space transport in their vicinity.

We check for the existence of transverse intersections between the stable and unstable manifolds $W^s(\mathcal{S}_\varepsilon)$ and $W^u(\mathcal{S}_\varepsilon)$ by using the Melnikov method. In particular, if we denote by $(p^h(\tau,z), q^h(\tau,z))$ any solution on the unperturbed homoclinic orbit $\Gamma(z)$, the *Melnikov function*

$$M(z) = \int_{-\infty}^{\infty} \left(\frac{\partial H}{\partial z}(p^h(\tau,z), q^h(\tau,z), z) - \frac{\partial H}{\partial z}(P(z), Q(z), z) \right) d\tau \qquad (3)$$

measures the energy difference between appropriately chosen orbits on the manifolds $W^s(\mathcal{S}_\varepsilon)$ and $W^u(\mathcal{S}_\varepsilon)$. (The original references for the Melnikov method are [2–4], see also the review in [5].) In particular, if $M(z)$ has a simple zero for some value of z, it follows that the corresponding unperturbed homoclinic orbit $\Gamma(z)$ will survive for sufficiently small nonzero ε, and that the manifolds $W^s(\mathcal{S}_\varepsilon)$ and $W^u(\mathcal{S}_\varepsilon)$ will intersect transversely along the surviving orbit. We remark that the Melnikov function $M(z)$ is periodic with period 2π in z, and, therefore, if it passes through zero once, it must do so infinitely many times.

One interesting fact that we want to highlight in this review is that formula (3) can be evaluated explicitly, without any knowledge of the solution $(p^h(\tau,z), q^h(\tau,z))$, but only on the basis of the geometry of the unperturbed system (2). In particular, we have the following

Theorem 1 *The Melnikov function $M(z)$ is the derivative of the area $A(z)$ enclosed by the homoclinic orbit $\Gamma(z)$ with respect to the parameter z.*

An immediate consequence of this theorem is

Corollary 1 *The Melnikov function has simple zeros at those values of z for which the area $A(z)$ enclosed by the homoclinic orbit $\Gamma(z)$ is locally a minimum or a maximum.*

Results that connect Smale-horseshoe-type chaotic dynamics and transverse homoclinic orbits, presented for instance in [6, 7], now imply that

Corollary 2 *If the area $A(z)$ enclosed by the homoclinic orbits $\Gamma(z)$ is not a constant function of z, then the system possesses transverse homoclinic orbits, and is hence chaotic.*

This corollary implies, in particular, that generic adiabatically and periodically perturbed planar Hamiltonian systems that satisfy assumption 1 exhibit Smale-horseshoe-type chaotic dynamics.

3 Lobe Area

In order to discuss transport of points in the phase space, which is addressed by the second result reviewed in this paper, we recall the definition of the Poincaré return map. This map takes every point on the $t = 0$ ($z = 0$) slice of the $p - q - t$ phase space to a point on the $t = \frac{2\pi}{\varepsilon}$ ($z = 2\pi$) slice. Because of the periodicity of equations (1) in $\varepsilon t = z$, we can think of the Poincaré map as a function mapping the $p - q$ plane into itself. Then, the intersections of the orbit $\mathcal{S}_\varepsilon$ and its stable and unstable manifolds $W^s(\mathcal{S}_\varepsilon)$ and $W^u(\mathcal{S}_\varepsilon)$ with the $p - q$ plane at $t = 0$ produce a fixed point X_ε and its stable and unstable manifolds $W^s(X_\varepsilon)$ and $W^u(X_\varepsilon)$ for the Poincaré map. If the manifolds $W^s(\mathcal{S}_\varepsilon)$ and $W^u(\mathcal{S}_\varepsilon)$ intersect then so must the curves $W^s(X_\varepsilon)$ and $W^u(X_\varepsilon)$.

We call a point of intersection of the curves $W^s(X_\varepsilon)$ and $W^u(X_\varepsilon)$ a homoclinic point. Let R be such a homoclinic point. If the segment of $W^s(X_\varepsilon)$ between X_ε and R and the segment of $W^u(X_\varepsilon)$ between X_ε and R only intersect at the points X_ε and R, we call R a primary intersection point. If P and Q are two adjacent primary intersection points, that is, there are no other primary intersection points on the segments of $W^s(X_\varepsilon)$ and $W^u(X_\varepsilon)$ connecting P and Q, then the area enclosed by these two segments is called a *lobe*. Lobes provide the mechanism for transport of phase points through the homoclinic tangle, that is, the tangling structure of the intersecting manifolds $W^s(X_\varepsilon)$ and $W^u(X_\varepsilon)$.

In adiabatic problems, lobes are usually whiskered objects such as the one shown in Figure 2. This can be shown by a careful examination of the parametrization of the manifolds $W^s(\mathcal{S}_\varepsilon)$ and $W^u(\mathcal{S}_\varepsilon)$ and the Melnikov function $M(z)$, by the usual Gronwall estimates, and by Kaplun's extension theorem described in [8]. Moreover, numerical results of [9, 10] have indicated that the area A_L of a given lobe L may be an $\mathcal{O}(1)$ quantity, and have evaluated it for the special case of the parametrically forced pendulum.

The next theorem rigorously establishes the lobe-area result in the class of systems under our consideration:

Theorem 2 *The area A_L of the lobe L is given by the formula*

$$A_L = \int_{z_1}^{z_2} M(z)dz + \mathcal{O}(\varepsilon) \tag{4}$$

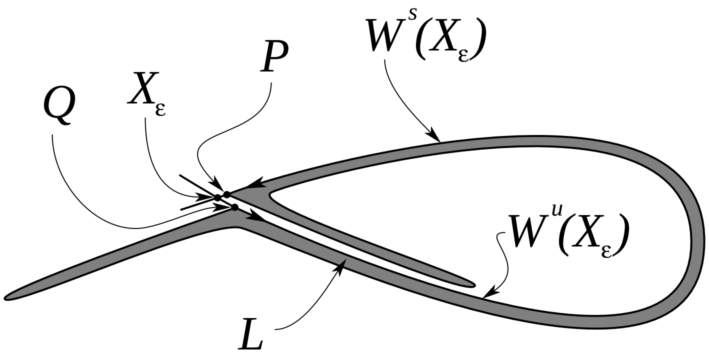

Figure 2: A lobe.

where z_1 and z_2 are the zeros of the Melnikov function $M(z)$ corresponding to the homoclinic orbits that pass through the end points of the lobe L.

This theorem was also derived in [11], but the proof given in [1] and reviewed in this note is considerably simpler. Theorems 1 and 2 imply that the lobe area A_L is in the first approximation equal to the difference $A(z_2) - A(z_1)$ of the areas $A(z_2)$ and $A(z_1)$ enclosed by the two unperturbed separatrices whose z-coordinates are z_2 and z_1, respectively. Since it also follows from Corollary 1 that the Melnikov function passes through zero at the local maxima or minima of the area $A(z)$ enclosed by the unperturbed separatrices $\Gamma(z)$, it follows that

Corollary 3 *The lobe area A_L of any given lobe L is the difference of the areas $A(z_2)$ and $A(z_1)$ enclosed by two unperturbed separatrices $\Gamma(z_2)$ and $\Gamma(z_1)$, which, at least locally, enclose a maximum and a minimum area, respectively.*

In the rest of this paper, we describe the proofs of Theorems 1 and 2.

4 Proof of Theorem 1

We begin this proof by investigating the geometric meaning of the integral $\int_{z_0}^{z} M(\zeta)d\zeta$. By the definition (3) of the Melnikov function we have

$$\int_{z_0}^{z} M(\zeta)d\zeta$$

$$= \int_{z_0}^{z} d\zeta \int_{-\infty}^{\infty} \left[\frac{\partial H}{\partial \zeta}(p^h(\tau,\zeta), q^h(\tau,\zeta), \zeta) - \frac{\partial H}{\partial \zeta}(P(\zeta), Q(\zeta), \zeta)\right] d\tau$$

We immediately notice that the second term in the integrand may be rewritten as $\frac{dH}{d\zeta}(p^h(\tau,\zeta), q^h(\tau,\zeta), \zeta)$. This is done by first noticing that, by continuity, we must have

$$H(p^h(\tau,\zeta), q^h(\tau,\zeta), \zeta) - H(P(\zeta), Q(\zeta), \zeta) = 0$$

for all τ and ζ, and therefore,

$$\frac{dH}{d\zeta}(p^h(\tau,\zeta), q^h(\tau,\zeta), \zeta) - \frac{dH}{d\zeta}(P(\zeta), Q(\zeta), \zeta) = 0.$$

Moreover,

$$\begin{aligned}
\frac{dH}{d\zeta}(P(\zeta), Q(\zeta), \zeta) &= \frac{\partial H}{\partial p}(P(\zeta), Q(\zeta), \zeta)\frac{dP}{d\zeta}(\zeta) \\
&\quad + \frac{\partial H}{\partial q}(P(\zeta), Q(\zeta), \zeta)\frac{dQ}{d\zeta}(\zeta) + \frac{\partial H}{\partial \zeta}(P(\zeta), Q(\zeta), \zeta) \\
&= \frac{\partial H}{\partial \zeta}(P(\zeta), Q(\zeta), \zeta)
\end{aligned}$$

since $(P(\zeta), Q(\zeta), \zeta)$ is an equilibrium point of the unperturbed system (2) for every ζ, which proves our claim.

The integral of the Melnikov function now reads

$$\begin{aligned}
&\int_{z_0}^{z} M(\zeta) d\zeta \\
&\quad = \int_{z_0}^{z} d\zeta \int_{-\infty}^{\infty} \left(\frac{\partial H}{\partial \zeta} - \frac{dH}{d\zeta} \right) (p^h(\tau,\zeta), q^h(\tau,\zeta), \zeta)\, d\tau \\
&\quad = -\int_{z_0}^{z} d\zeta \int_{-\infty}^{\infty} \left(\frac{\partial H}{\partial p}\frac{\partial p^h}{\partial \zeta} + \frac{\partial H}{\partial q}\frac{\partial q^h}{\partial \zeta} \right) (p^h(\tau,\zeta), q^h(\tau,\zeta), \zeta)\, d\tau \\
&\quad = \int_{z_0}^{z} d\zeta \int_{-\infty}^{\infty} \left(\frac{\partial p^h}{\partial \tau}\frac{\partial q^h}{\partial \zeta} - \frac{\partial q^h}{\partial \tau}\frac{\partial p^h}{\partial \zeta} \right) d\tau \\
&= \iint_{\Omega(z,z_0)} dp \wedge dq,
\end{aligned} \tag{5}$$

where $\Omega(z, z_0)$ is the tubular surface spanned by the homoclinic orbits $(\Gamma(\zeta), \zeta)$ for ζ between z_0 and z, shown in Figure 3. Alternatively, the integral $\int_{z_0}^{z} M(\zeta)d\zeta$ is the area of the projection of the surface $\Omega(z, z_0)$ onto the $p - q$ plane.

Now, it can be easily seen by using Stokes' theorem that we can deform the surface $\Omega(z, z_0)$ into any other surface suspended between the two homoclinic orbits $(\Gamma(z), z)$ and $(\Gamma(z_0), z_0)$, and the projected area will stay the same. In particular, we can deform the

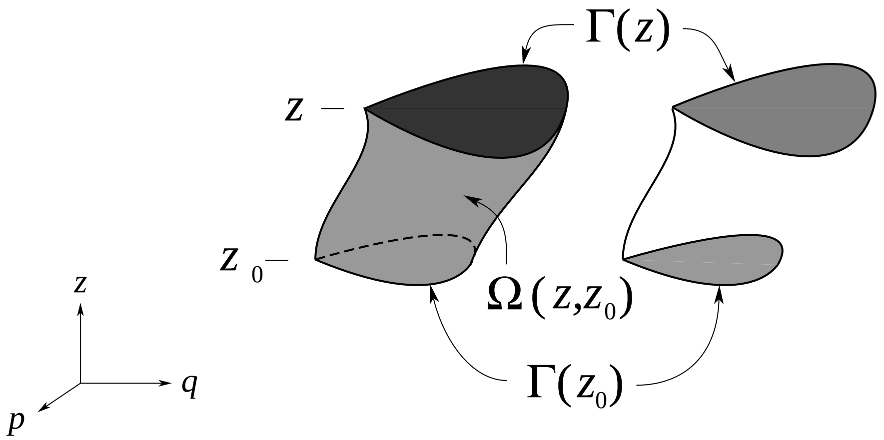

Figure 3: The surface $\Omega(z, z_0)$.

surface $\Omega(z, z_0)$ into the degenerate surface that consists of the two planar surfaces enclosed by the separatrices $(\Gamma(z), z)$ and $(\Gamma(z_0), z_0)$ on the two horizontal planes $\zeta = z$ and $\zeta = z_0$, respectively, and the connecting curve $(P(\zeta), Q(\zeta), \zeta)$ with ζ between z_0 and z, which is also shown in Figure 3. But the area of the projection of this surface onto the $p - q$ plane is precisely the difference $A(z)-A(z_0)$ of the areas $A(z)$ and $A(z_0)$ enclosed by the separatrices $\Gamma(z)$ and $\Gamma(z_0)$ in the $p - q$ plane, respectively.

Differentiating formula (5), we conclude that $M(z) = dA(z)/dz$. That is, the Melnikov function $M(z)$ is the derivative with respect to z of the area $A(z)$ enclosed by the unperturbed separatrix $\Gamma(z)$. This implies Theorem 1.

5 Proof of Theorem 2

In order to derive formula (4), we recall that the area A_L is given by the formula

$$A_L = \iint_L dp \wedge dq.$$

Consider now a vertical plane Π in the $p - q - z$ space intersecting all the trajectories of points in the lobe L. The image of the lobe L in the plane Π is another lobe-like object, which we will call K, and is shown in Figure 4. Let the surface Σ be composed of the lobes L and K and the surface spanned by the trajectories connecting the boundary ∂L of the lobe L to the boundary ∂K of the lobe K. On the surface Σ, consider the integral [12]

$$\iint_\Sigma dp \wedge dq - dH \wedge dt. \tag{6}$$

On L, the second part of the integral vanishes since L lies on a constant t-slice. On K, the first part of the integral vanishes since K is vertical and thus its projection into the $p - q$

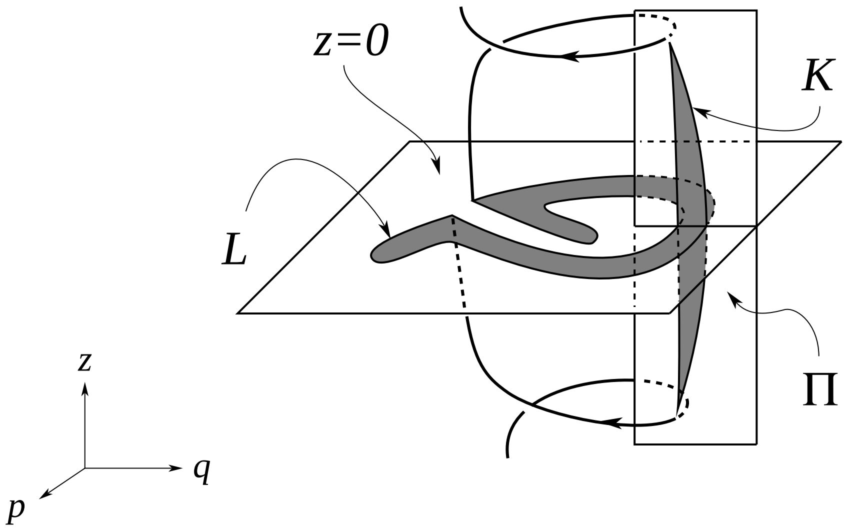

Figure 4: The lobes L and K.

plane is a line segment with zero area. And finally, on the surface spanned by orbits, the whole integral vanishes since it measures the flux of phase points through that surface, which is equal to zero. (Alternatively, the differential form $dp \wedge dq - dH \wedge dt$ applied to any pair of vector fields, one of which is the vector-field (1), gives identically zero result.)

By Stokes' theorem, the integral (6) is equal to the integral of the differential form $d(dp \wedge dq - dH \wedge dt)$ over the volume enclosed by the surface Σ. But since $dp \wedge dq - dH \wedge dt = d(pdq - Hdt)$, its differential is zero, and hence

$$\iint_\Sigma dp \wedge dq - dH \wedge dt = \iint_L dp \wedge dq - \iint_K dH \wedge dt = 0. \tag{7}$$

Here, the orientation of the lobe K must be the same as the orientation of the lobe L. Again, by Stokes' theorem, the integral $\iint_K dH \wedge dt$ is equal to the integral $\int_{\partial K} Hdt$ with the appropriately chosen orientation of the boundary ∂K. If we denote by ∂K^s the part of the boundary ∂K that is contained in the manifold $W^s(\mathcal{S}_\varepsilon)$ and by ∂K^u the part of ∂K contained in the manifold $W^u(\mathcal{S}_\varepsilon)$, we finally obtain

$$A_L = \int_{\partial K^u} Hdt - \int_{\partial K^s} Hdt.$$

If $c^s_\varepsilon(t)$ and $c^u_\varepsilon(t)$ denote two points on ∂K^s and ∂K^u with the same t-coordinate (or equivalently, the same z-coordinate), then

$$A_L = \int_{t_1}^{t_2} \left[H(c^u_\varepsilon(t)) - H(c^s_\varepsilon(t))\right] dt,$$

where t_1 and t_2 are t coordinates of the end points of the lobe K, the points $c^s_\varepsilon(t_1) = c^u_\varepsilon(t_1)$ and $c^s_\varepsilon(t_2) = c^u_\varepsilon(t_2)$, respectively.

Now, the standard Melnikov theory, as reviewed for instance in [5], implies that $H(c_\varepsilon^u(t)) - H(c_\varepsilon^s(t)) = \varepsilon M(\varepsilon t) + \mathcal{O}(\varepsilon^2)$. Therefore,

$$\begin{aligned} A_L &= \int_{t_1}^{t_2} [\varepsilon M(\varepsilon t) + \mathcal{O}(\varepsilon^2)] dt = \\ &= \int_{\varepsilon t_1}^{\varepsilon t_2} [M(z) + \mathcal{O}(\varepsilon)] dz = \\ &= \int_{z_1}^{z_2} M(z) dz + \mathcal{O}(\varepsilon), \end{aligned}$$

where, as mentioned before, $z_1 = \varepsilon t_1 + \mathcal{O}(\varepsilon)$ and $z_2 = \varepsilon t_2 + \mathcal{O}(\varepsilon)$ are the zeros of the Melnikov function $M(z)$ corresponding to the end points of the lobe K. This proves Theorem 2.

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