Linear Algebra, MATH-410

Test #1

Due: In class, on Monday, October 4, 1999

Instructor: Gregor Kovačič

NAME:___________________________

- Try to do the test in one continuous sitting. The test should not take you more than two hours, but you may run over as much as you need to if necessary.
- You may not use any textbooks, notes, calculators, or other aids, except for one, two-sided, $8\frac{1}{2} \times 11$ inch crib sheet.
- Each of the five main problems is worth 20 points. The sixth, extra credit, problem is worth 10 points.
- Show all your work, and explain your reasoning as well as you can.

PROBLEM #	POINTS
1	
2	
3	
4	
5	
6	
Total	

1. Determine values of k so that the system

$$\begin{array}{rcrcrcl} kx & + & y & + & z & = & 1 \\ x & + & ky & + & z & = & 1 \\ x & + & y & + & kz & = & 1 \end{array}$$

has (a) a unique solution, (b) no solution, (c) infinitely many solutions.

2. (a) Find an orthonormal basis for the orthogonal complement of the linear span of the vectors $(1, 0, 1, 0)$ and $(1, 0, -1, 0)$ in $\mathbb{R}^4$.

(b) Find two bases of $\mathbb{R}^3$ from among the vectors $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$, $(1, -1, 0)$ which have at most one vector in common.

3. (a) Suppose U and W are 2-dimensional subspaces of $\mathbb{R}^3$. Show that $U \cap W \neq 0$. In particular, find the possible dimensions of $U \cap W$. Interpret the results geometrically.

(b) Let U be the linear span of the vectors $(1, 2, 1)$ $(2, -2, 3)$, $(-3, 0, -4)$, and V the linear span of the vectors $(4, 2, 5)$ and $(1, 0, -1)$. Find a basis for U, one for V and one for their $U + V$. What is the dimension of the intersection $U \cap V$?

4. Let A be the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Show that the expression $\langle x, y \rangle = (x, Ay)$ defines an inner product on $\mathbb{R}^2$. Here $(u, v) = u_1 v_1 + u_2 v_2$ denotes the usual inner product.

5. The linear transformation $A : \mathcal{P}_2 \to \mathcal{P}_2$ is given by $Ap(t) = p(t+1)$. Find its matrix in the basis 1, t, t^2.

6. Extra Credit: Let A be a linear operator $A : V \to V$.

(a) Show that $A^2 = 0$ if and only if $\operatorname{Im} A \subset \operatorname{Ker} A$.

(b) Show that $\operatorname{Ker} A \subset \operatorname{Ker} A^2 \subset \operatorname{Ker} A^3 \subset \cdots$ and $\cdots \subset \operatorname{Im} A^3 \subset \operatorname{Im} A^2 \subset \operatorname{Im} A$.