Linear Algebra, MATH–410 Gregor Kovačič

Homework #7 Due: Wednesday, October 20, 1999

1. Compute the determinant

$$\begin{vmatrix} 2 & 3 & -1 & -2 \\ 3 & -4 & 2 & 1 \\ 2 & 2 & -3 & 1 \\ -8 & 5 & -1 & -4 \end{vmatrix}.$$

2. By induction, compute the Van Der Monde determinant of order n

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}.$$

HINT: Multiply the $(j-1)$-st row by λ_j and subtract it from the j-th row. Expand by the resulting first column, whose all elements except for the first are zero. In each column of the only minor that is left, all the elements should contain a common factor. After factoring out all these factors, you should be left with the Van Der Monde determinant of order $(n-1)$.

3. Compute the determinant

$$\begin{vmatrix} -1 & 1 & \dots & 1 \\ 1 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -1 \end{vmatrix}.$$

HINT: Add all the rows to the last. Factor out a common factor from the resulting last row, which will then become a row of 1's. Subtract this row from all the previous rows.

4. Without expanding the determinant, show that

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = 0.$$

HINT: Add the second column to the third.

5. Find the volume of a parallelepiped spanned by the vectors $(2,5,2)$, $(4,2,3)$, and $(1,1,4)$.

6. Show that the vector product $\mathbf{x} \times \mathbf{y}$ in $\mathbb{R}^3$ can be written in the form of the symbolic determinant

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, and $\mathbf{k} = (0,0,1)$.

7. Show that the oriented volume in $\mathbb{R}^3$ of the parallelepiped spanned by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ equals

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

8. Extra Credit: Let A be an $n \times n$ real matrix. Show that the volume of the parallelepiped spanned by the columns of A equals $\sqrt{\det A^T A}$. If the columns of the matrix A are $A_1, \ldots A_n$, what are the elements of the matrix $A^T A$?